Kontinuumsmechanik[1]

Prof. Geshkenbein, FS 2019
August 29, 2019, 11:00-11:30, Janik Schüttler

Summary The exam took place in Prof Geshkenbein's office. It was on paper and I was offered a huge choice of pens. The table was quite small and had to fit the assistant (Tudor Pahomi), Prof. Geshkenbein, and me. The assistant took notes as far as I noticed. Geshkenbein told us that he will ask questions from the exercises so that people will be motivated to solve them.

Description of the content: Hooke's law, homogenous deformation, Euler's equation, shape of water in rotated cylinder (exercise 7.3 from the problem sheets), Reynolds number.

Ablauf Prof. Geshkenbein asked me in, I shook hands with him and the assistant. Prof Geshkenbein then disappeared for 5 minutes and the assistant and me just sat there and chatted a little. When Prof. Geshkenbein came back, he asked me for my name and started the exam right away.

Prof: Can you derive Hooke's law?

Me: Sure. We start by defining the free energy. From a previous derivation we know that $\sigma_{ik} = \frac{\partial F}{\partial u_{ik}}\Big|_T$ which means that free energy must be a function of the strain tensor u_{ik}. Since in equilibrium there should be no stresses there will be no linear strain terms in the free energy if we expand in strain. For isotropic bodies we write down the following quadratic combination

$$F = F_0 + \frac{1}{2}\lambda(u_{ll})^2 + \mu u_{ik}u_{ik}.$$

Prof: Can you write down u_{ik}?

Me: It is defined as the symmetric combination of position (*yea, I said position, that was a little blackout lol even though this is probably the easiest question he could have asked*)

$$u_{ik} = \frac{1}{2}\left(\frac{\partial r_i}{\partial x_k} + \frac{\partial r_k}{\partial x_i}\right).$$

Prof: What is this r (*of course noticing my blackout*)?

Me: (*Remembering*) The deformation. We usually write it as u_i.

Prof: Okay. Why is the free energy a function of this strain? Why do we need a tensor? Why would we want a symmetric tensor in the free energy? Why can't we write it differently? (*He took maybe one minute to ask all of these questions with many uuhms and pauses, so I didn't know exactly what he wanted me to answer.*)

[1] BSc Physics, Core Subjects

Me: (*I tried to answer all of the questions one by one.*) The free energy should be a function of the strain because of this derivation...

Prof: (*Interrupting*) ...yes, but why do we want a symmetric tensor? Why can't it for example be $u_{ik} = \frac{\partial u_i}{\partial x_k}$ or $u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right)$?

Me: The idea is that the free energy should be invariant under rotations. Symmetric tensors are rotation invariant and that's why we use a symmetric tensor to write down the free energy. The strain itself is symmetric by convention.

Prof: (*Not fully satisfied with my answer*). But can't we just use a non-symmetric version of the strain?

Me: (*I ran out of arguments, so I tried something else*). Every non-symmetric tensor can be decomposed into a symmetric and an antisymmetric part. The antisymmetric part does not matter under rotations so we can safely neglect it and the symmetric part remains.

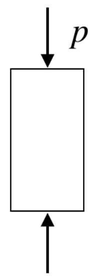
Prof: Okay. (*He seemed satisfied now, but not totally sure. He went on to explain something, which I don't recall*). How would you now derive Hooke's law from this?

Me: We use this formula to get a relation between the stress and the strain. By taking the derivative of the free energy we obtain Hooke's law in its most basic form which reads

$$\sigma_{ik} = \frac{\partial F}{\partial u_{ik}} = \lambda u_{ll}\delta_{ik} + 2\mu u_{ik},$$

but there are other variants using different constants.

Prof: Imagine the following situation: we have a body where we apply a pressure p to two sides. What will happen? (*He draws the following sketch similar to the homogenous deformation example in lecture 2.*)



Me: The body will be compressed in one direction, let's call this direction z, and widened in the orthogonal directions. Quantitatively we require the boundary condition $\sigma_{zz} = p$ and all other components of the stress to be 0. From a different representation of Hooke's law we find that $u_{zz} = \frac{\sigma_{zz}}{E} = \frac{p}{E}$, where E is the compression modulus.

Prof: And how would you derive the other components?

Me: Using Poisson's ratio σ we can relate the strain in z direction to the orthogonal strains in x and y direction. It is defined such that for homogenous deformations in one direction the orthogonal strain components can be computed as $u_{xx} = u_{yy} = -\sigma u_{zz}$.

Prof: Okay. How would this problem change if the body was inclined between two non-deformable walls? Would the deformation be larger or smaller?

Me: The deformation in x and y direction would be zero, so definitely smaller then in the previous problem...

Prof: (*Interrupting*) ...yea, but in the z direction?

Me: There the compression would be less. If we assume the same elasticity for both settings, the body cannot be deformed in x and y direction anymore. We would model this problem using the additional boundary conditions that the deformation in x and y direction must vanish, i.e. $u_x = u_y = 0$.

Prof: But cannot you see directly what would change?

Me: Mhm, I mean we see what would happen from this, don't we?

Prof: In the free energy only the u_{zz} strain term would remain and all others would vanish so Hooke's law would reduce to only this term and we see directly the relation.

Me: Mhm I see, but I would argue that this is somewhat the same approach. Using the boundary conditions is only the more formal way of treating this problem, in the end Hooke's law will also reduce to this using these boundary conditions.

Prof: Can you derive Euler's equation?

Me: Sure. Euler's equation is a fancy way of stating Newton's second law. We first write down the force as the pressure acting on a surface of a volume of the fluid and we usually give it a minus to account for the inward force

$$\mathbf{F} = -\oint p \,\mathrm{d}\mathbf{S} = -\int \nabla p \,\mathrm{d}V,$$

where I've used Gauss' theorem. On the other hand, using Newton's second law explicitly, we can write the force as

$$\mathbf{F} = m\mathbf{a} = \int \rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \,\mathrm{d}V$$

Equating both and using the facts that this must hold for all unit volumes we get the following equality

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \nabla p.$$

Now the time derivative of the velocity has a little subtlety. Since it is a function of position and time and position itself is a function of time, we have to employ the chain rule to calculate it. Using the chain rule, we arrive at $\frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{\partial v_i}{\partial t} + \frac{\mathrm{d}x_k}{\mathrm{d}t}\frac{\mathrm{d}v_i}{\mathrm{d}x_k}$, which again in vector notation reads

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}.$$

Plucking this into the above equation yields Euler's equation.

Prof: Which is?

Me: Usually we take the density to the other side and obtain the following

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho}.$$

Prof: Can you apply Eulers equation to a rotating cylinder to calculate the shape of the surface?

Me: (*This was exercise 7.3 from problem sheet 7*). Yes, the equilibrium condition states that the gradient of pressure must equal external forces $\mathbf{F} = \nabla p$. This is something I forgot to mention in the previous derivation. From Newtons second law we see that external forces can be added to Eulers equation (*Geshkenbein makes a sign that I should go on*). The external force in our case are gravity and the centrifugal force. If we write down the z and r components of the equilibrium equation we get

$$\partial_r p = \rho\omega^2 r,$$
$$\partial_z p = -\rho g.$$

From this we find the pressure by integrating

$$p(z, r) = \frac{\rho}{2}\omega r^2 - \rho g z + const.$$

with some constant that is of no interest. On the surface pressure must balance atmospheric pressure...

Prof: (*interrupting*) ...yes, but what is the shape of the water?

Me: So the short story is that using this equation we see that the relation between the radius r and the height z is quadratic which shows that the shape of the water must be quadratic in the radius.

Prof: Good. What if we stir a cup of tea. Can you estimate Reynolds number for this situation?

Me: I would guess it is relatively high (*I know the answer would be around 100, because he mentioned it once in a lecture*). The Reynolds number is defined as the ratio of velocity, body size and viscosity, $\mathrm{Re} = \frac{vL}{\nu}$. I am not sure about the magnitude of the viscosity of water though...

Prof: In which units do you need it?

Me: The normal units of meters and seconds.

Prof: Use 10^{-6}.

Me: Okay. For the body size we can assume a magnitude of say 10^{-1}, while for the velocity we get something like maybe $1 - 10^1$ m/s.

Prof: (*Prof and assistant look at each other and make movements that this would be super fast*).

Me: I agree, lets maybe use something in the range of $10^{-2} - 10^{-3}$, but it doesn't really matter, because what we see is that Reynolds number nevertheless will be large and of order ~ 100.

Prof: Does a Reynolds number of such magnitude affect the problem?

Me: We usually have two limits, the high Reynolds number limit for Reynolds numbers above around 40 and the low Reynolds number limit for Reynolds numbers far smaller than 1, so this would be in the very non-viscous regime.

Prof: But would the problem change if we used a viscous fluid instead of a non-viscous fluid?

Me: In the equation you would get an additional term, a Laplacian, but my feeling is that the equilibrium condition $\nabla p = F$ would not change in such a regime (*I take at least 15 seconds to think about it*), although I am not sure why.

Prof: Using the same arguments about rotation from the beginning one could see that the Laplacian will vanish. One could of course remark that this symmetry might not be super stable as we estimated it for the river flow, but in principle this would be the answer.

Me: Right, that makes sense (*it really did and I wanted to talk more about it, but time was up*).

Prof: Good, we are done.

Final Remarks I feel like Geshkenbein and me sometimes thought of different things, meaning I often felt I understood his questions not like he did and did not answer what he wanted to hear. He interrupted me from time to time which I found somewhat confusing. After I left the exam I was a bit angry about myself for all my tiny unnecessary mistakes. However, these mistakes seemed to not have influenced his grading. Together with my impression of him often not being satisfied with my answers I though I would be graded a good part below 6.

Expected mark: 5
Received mark: 6