Mechanics of Continua

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Contents

1 Catenary, Suspension Bridge and Elastic String

Equilibrium shapes of hanging inelastic cable, a suspension bridge and an elastic string.

$$u(x) = \frac{\boldsymbol{T}_x}{\rho g}\left(\cosh\left(\frac{\rho g}{\boldsymbol{T}_x}x\right) - 1\right) \quad \text{(Catenary)}$$

$$u(x) = \frac{\rho h g}{2\boldsymbol{T}_x}x^2 \quad \text{(Suspension Bridge)}$$

$$u(x) = \frac{\rho h g}{2\boldsymbol{T}_0}x^2 \quad \text{(Elastic String)}$$

Derivation Catenary $\boldsymbol{T}(x+\mathrm{d}x) - \boldsymbol{T}(x) = \rho\boldsymbol{g}\,\mathrm{d}l$ and suspension bridge $\boldsymbol{T}(x+\mathrm{d}x) - \boldsymbol{T}(x) = \rho\boldsymbol{g}\,\mathrm{d}x$. Rewrite $\mathrm{d}l$. $T_x = const$. Tension is tangential to line $\frac{T_y}{T_x} = \frac{\mathrm{d}u}{\mathrm{d}x}$. Solve using $v = u'$. Tension T_x from boundary condition (chain length). For elastic string minimize energy $\delta E = T_0\delta L + E_{gravitation} = T_0\int\sqrt{1+u'^2} - 1\,\mathrm{d}x + \int\rho g u\,\mathrm{d}x$. Use Taylor for first expression.

2 Elasticity Theory

2.1 Strain and Stress Tensors

Displacement vector $\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{r}' - \boldsymbol{r}$.

Strain tensor is the symmetric tensor

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_i}\right) \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

Rel change of volume $\frac{\mathrm{d}V' - \mathrm{d}V}{\mathrm{d}V} = u_{ii} = \operatorname{div}\boldsymbol{u}$

Derivation $\mathrm{d}l' = (\mathrm{d}x_i + \mathrm{d}u_i)^2 = \ldots = \mathrm{d}l^2 + 2u_{ik}\,\mathrm{d}x_i\,\mathrm{d}x_k$ by expanding and using $\mathrm{d}u_i = \frac{\partial u_i}{\partial x_k}\,\mathrm{d}x_k$. Assume small displacement, then $\frac{\partial u_i}{\partial x_k} \ll 1$ and quadratic terms can be neglected. Volume $\mathrm{d}V' = \mathrm{d}x_1'\,\mathrm{d}x_2'\,\mathrm{d}x_3' = \mathrm{dV}(1+u_{11})(1+u_{22})(1+u_{33}) = \mathrm{dV}(1+u_{ii}+\ldots)$.

Shear and compression The strain tensor can be rewritten as $u_{ik} = (u_{ik} - \frac{1}{3}\delta_{ik}u_{ll}) + \frac{1}{3}\delta_{ik}u_{ll}$. The first part is called shear (only off-diagonal) and corresponds to volume perserving deformations, the second part is called compression (only diagonal) and corresponds to shape perserving deformations..

Stress tensor σ_{ik}

$$F_i = \frac{\partial\sigma_{ik}}{\partial x_k} \quad \text{(Stress Tensor)}$$

Derivation Newton's third law: total inner force from the inner part is zero, hence all forces arise at the surface.

Stress and energy $\sigma_{ik} = \left(\frac{\partial F}{\partial u_{ik}}\right)_T = \left(\frac{\partial U}{\partial u_{ik}}\right)_S$

Derivation Calculate work $\delta w = F_i\delta u_i = \frac{\partial\sigma_{ik}}{\partial x_k}\delta u_i$ in a volume integral, use partial integration B.1, taking surface to infinity trick B.2. Calculate $\mathrm{d}U = T\,\mathrm{d}s - \delta w = T\,\mathrm{d}s + \sigma_{ij}\,\mathrm{d}u_{ij}$ and $\mathrm{d}F = -s\,\mathrm{d}T + \sigma_{ik}\,\mathrm{d}u_{ik}$.

Moment of forces (torque)

$$M_{ik} = \oint \sigma_{il}x_k - \sigma_{kl}x_i\,\mathrm{d}\boldsymbol{S}_l \quad \text{(Torque)}$$

Derivation Use $M_{ik} = \int(F_i x_k - F_k x_i)\,\mathrm{dV}$, $F_i = \frac{\partial\sigma_{ik}}{\partial x_k}$ reverse product rule B.3, Gauss' theorem $\int\frac{\partial A}{\partial x_l} = \oint A\,\mathrm{d}S_l$ and symmetry of σ_{ik}.

Constants used in subsequent equations.

- *Lamé Coefficients* λ, μ with $\mu > 0$. $\lambda > 0$ holds in practice, but not required from thermodynamics.
- *Compression Modulus* $K = \lambda + \frac{2}{3}\mu > 0$
- *Young Modulus* $E = \frac{9K\mu}{3K+\mu}$, also coefficient of extension.
- *Poisson's Ratio* $\sigma = \frac{1}{2}\frac{3K-3\mu}{3K+2\mu}$ is the ratio of the transverse compression to the longitudinal extension. Theoretically $-1 \le \sigma \le 1/2$, experimentally $0 \le \sigma \le 1/2$

2.2 Boundary Conditions

Hydrostatic compression w condition $-p\,\mathrm{d}\boldsymbol{S}_i = -p\delta_{ik}\,\mathrm{d}\boldsymbol{S}_k$ yields BC $\sigma_{ik} = -p\delta_{ik}$.

External force at surface with condition $P_i\,\mathrm{d}S = \sigma_{ik}\,\mathrm{d}\boldsymbol{S}_k = \sigma_{ik}n_k\,\mathrm{d}S$ yields BC $\sigma_{ik}n_k = P_i$.

2.3 Hooke's Law

Equilibrium state satisfies $\sigma_{ik} = u_{ik} = 0$.

Free Energy per unit volume

$$f = f_0 + \frac{\lambda}{2}(u_{ii})^2 + \mu u_{ik}u_{ik} = \frac{1}{2}\sigma_{ik}u_{ik}$$

$$f = \frac{E}{2(1+\sigma)}\left(u_{ik}^2 + \frac{\sigma}{1-2\sigma}u_{ll}^2\right)$$

Derivation *Alternative 1*: In equilibrium $\sigma_{ik} = 0$, hence F quadratic in u_{ik}. Neglect higher order terms. *Alternative 2*: Energy must depend on gradient of displacement, but must be rotation invariant and hence should not contain the antisymmetric part $\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}$ terms.

Hooke's Law

$$\begin{aligned}\sigma_{ik} &= \lambda u_{ll}\delta_{ik} + 2\mu u_{ik} = K u_{ll}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{ll}) \\ &= \frac{E}{1+\sigma}\left(u_{ik} + \frac{\sigma}{1-2\sigma}u_{ll}\delta_{ik}\right) \\ u_{ik} &= \frac{1}{9K}\delta_{ik}\sigma_{ll} + \frac{1}{2\mu}(\sigma_{ik} - \frac{1}{3}\delta_{ik}\sigma_{ll}) \quad \text{(Hooke's Law)} \\ &= \frac{1}{E}\left((1+\sigma)\sigma_{ik} - \sigma\delta_{ik}\sigma_{ll}\right)\end{aligned}$$

Derivation Vary F with respect to u_{ik} and invert expression to obtain Hook's law.

2.4 The Equation of Equilibrium for isotropic Bodies

Homogeneous Deformations are deformations where the strain tensor is constant throughout the volume of the body.

Equilibrium equation

$$\begin{aligned}\mu\nabla^2\boldsymbol{u} + (\mu+\lambda)\,\text{grad div}\,\boldsymbol{u} &= -\boldsymbol{F} \\ \mu\frac{\partial^2 u_i}{\partial x_k^2} + (\mu+\lambda)\frac{\partial^2 u_l}{\partial x_i \partial x_l} &= -F_i\end{aligned}$$

Alternative representations include

$$\begin{aligned}\frac{E}{2(1+\sigma)}\frac{\partial^2 u_i}{\partial x_k^2} + \frac{E}{2(1+\sigma)(1-2\sigma)}\frac{\partial^2 u_l}{\partial x_i \partial x_l} &= -\rho g_i \\ \Delta\boldsymbol{u} + \frac{1}{1-2\sigma}\,\text{grad div}\,\boldsymbol{u} &= -\rho\boldsymbol{g}\frac{1+\sigma}{E} \\ \frac{2-2\sigma}{1-2\sigma}\,\text{grad div}\,\boldsymbol{u} - \text{rot rot}\,\boldsymbol{u} &= -\rho\boldsymbol{g}\frac{1+\sigma}{E}\end{aligned}$$

Derivation The equilibrium condition states $0 = \sum F = \frac{\partial \sigma_{ik}}{\partial x_k} + F_{i,ext}$. Use $F_{ext} = \rho g_i$ and rewrite terms.

2.5 Thermal Expansion

Free energy under thermal expansion $F(T) = F_0(T) - K\alpha(T-T_0)u_{ii} + \frac{1}{2}Ku_{ll}^2 + \mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{ii})^2$

Stress under thermal expansion $\sigma_{ik} = -K\alpha(T-T_0)\delta_{ik} + Ku_{ll}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{ll})$

Derivation For $T = T_0$ body undeformed, $T \neq T_0$ body will be deformed even without external forces, hence F becomes linear in $A(T)u_{ii}$. Taylor A around T_0 and keep only linear term.

Volume change from heating $\delta V/V = u_{ll} = \alpha(T-T_0)$ when there are no external forces. α is the thermal expansion coefficient.

Derivation For $\sigma_{ik} = 0$ we get $u_{ik} \propto \delta_{ik}$.

Equation of equilibrium for non-uniformly heated isotropic bodies

$$\text{grad div}\,\boldsymbol{u} - \frac{1-2\sigma}{2(1-\sigma)}\,\text{rot rot}\,\boldsymbol{u} = \alpha\nabla T$$

2.6 Elasticity of Crystals

Elastic modulus tensor is the tensor λ_{iklm} s.t.

$$F = \frac{1}{2}\lambda_{iklm}u_{ik}u_{kl}$$

and hence $\sigma_{ik} = \lambda_{iklm}u_{lm}$ holds. In general for isotropic bodies it is given by

$$\lambda_{iklm} = \lambda\delta_{ik}\delta_{lm} + \mu(\delta_{il}\delta_{km} + \delta_{im}\delta_{kl}), \quad \text{(Elastic Modulus)}$$

and has 21 independent components.

Derivation 6 independent combinations of $\{x, y, z\}$. First pair can be combined with 6 other pairs, second with 5 other pairs etc. $21 = 6+5+4+3+2+1$.

Monoclinic has 13 independent components.
Orthorombic has 9 independent components.
Tetragonal System has 6 independent components because of mirror and rotation symmetry. It consists of a cube with two sides of the same length.
Hexagonal has 5 independent components.
Cubic System has 3 independent components. It consists of a cube with three sides of the same length.

Thermal expansion $u_{ik} = \frac{1}{3}\alpha_{ik}(T-T_0)$ where α_{ik} is a symmetric tensor with varying number of components: 3 (triclinic, monoclinic, orthorombic), 2 (tetragonal), 1 (cubic).

Elastic energy of classical harmonic lattice

$$\delta E_{int} = \frac{N}{16}\sum[C_{ik}R_jR_l + C_{jk}R_iR_l + C_{il}R_jR_k + C_{jl}R_iR_k]u_{ij}u_{kl}$$

$$\lambda_{ijkl} = \frac{1}{8V_0}\sum C_{ik}(\boldsymbol{R})R_jR_l + C_{jk}(\boldsymbol{R})R_iR_l + C_{il}(\boldsymbol{R})R_jR_k + C_{jl}(\boldsymbol{R})R_iR_k$$

Derivation Start with $E_{int} = \frac{1}{2}\sum_{R,R'} V(\boldsymbol{R} + \boldsymbol{u}(\boldsymbol{R}) - (\boldsymbol{R}' + \boldsymbol{u}(\boldsymbol{R}')))$, Taylor, linear term vanishes, use $C_{ij} = \frac{\partial^2 V}{\partial R_i \partial R_j}$. Expand $u_i(\boldsymbol{R}) = u_i(\boldsymbol{R}') + \frac{\partial u_i}{\partial R_j}(R - R')_j$ and shift $\boldsymbol{R} - \boldsymbol{R}' \to \boldsymbol{R}$. Replace $\frac{\partial u_i}{\partial R_j}$ by u_{ij} (energy does not change under rotation). Add combinations by exchanging $i \leftrightarrow j$, $k \leftrightarrow l$, pull $u_{ij}u_{kl}$ out of the bracket and compare with $\delta E_{el} = \frac{1}{2}\int \lambda_{ijkl}u_{ij}u_{kl}\,\mathrm{dV}$ to obtain λ_{ijkl}. $V = NV_0$ where V_0 is the unit cell volume.

2.7 Bending of Rods

Assumptions Displacements are small, rod is thin, forces at surface to bend rod are small and can be neglected, rod parallel to x-axis.

Boundary conditions $\sigma_{ik}n_k = 0 = \sigma_{zz}n_z + \sigma_{zy}n_y$ for rod along x-axis (i.e. $n_x = 0$).

Components of σ_{ik} are all zero except for σ_{xx}

Derivation For some point on the circumference of the cross section $n_y = 0$ and then BC $\implies 0 = \sigma_{zz}n_z \implies \sigma_{zz} = 0$. Similarly for $\sigma_{yy} = 0$. Rod is thin, hence $\sigma_{zz} = \sigma_{yy} = 0$ everywhere.

Neutral surface passes through center of mass.

Derivation Internal stress force on a cross-section $\int \sigma_{xx}\,\mathrm{d}S = \int z\,\mathrm{d}S = 0$, which is the z coordinate of the center of mass.

Deformation for bent rod is

$$u_z = -\frac{1}{2R}(x^2 + \sigma(z^2 - y^2)), u_y = -\frac{\sigma zy}{R}, u_x = \frac{zx}{R}.$$

Derivation Length of neutral surface $\mathrm{d}x = R\,\mathrm{d}\varphi$, length away from neutral surface $\mathrm{d}x + \mathrm{d}u_x = (R + z)\,\mathrm{d}\varphi \implies u_{xx} = \frac{z}{R}$, $u_{yy} = u_{zz} = \sigma u_{xx}$, $\sigma_{xx} = Eu_{zz}$. Integrate to get u_x, u_y, from that construct u_z such that $u_{xz} = u_{xy} = u_{yz} = 0$.

Equation of equilibrium for a bent rod is

$$F_z = IEz^{(4)}. \qquad \text{(Eq of equilibrium)}$$

Its energy is $F = \int \frac{1}{2}IE(z'')^2 + U(z,x)\,\mathrm{d}x$, its torque $M_y = \frac{EI_y}{R}$.

Derivation Free energy: $f = \frac{1}{2}\sigma_{ik}u_{ik} = \frac{1}{2}\sigma_{xx}u_{xx}$. Use $\int z^2\,\mathrm{d}S = I_y$. Torque $M_y = \int \sigma_{xx}z\,\mathrm{d}S$. Rewrite F using $1/R = \pm\frac{\mathrm{d}^2z}{\mathrm{d}x^2}$. Add potential $U(z,x)$. Vary with respect to z to obtain equation of equilibrium. Opt: For bends in z and y direction add deriv to F.

2.8 Applications: Examples of Deformations

2.8.1 Rod bent by its own Weight

Boundary conditions for bent rod

- *clamped* $z = 0$, $z' = 0$
- *supported* $z = 0$, $z'' = 0$ (torque is zero)

Equation for a rod either clamped on one side or supported on both sides

$$z = \frac{\rho g}{24EI}x^2(x - L)^2 \qquad \text{(clamped, 2 sides)}$$

$$z = \frac{\rho g}{24IE}x(x^3 - 2Lx^2 + L^3) \qquad \text{(supported, 2 sides)}$$

$$z = \frac{f}{6EI}x^2(3L - x) \qquad \text{(clamped, 1 side)}$$

Derivation Use Ansatz $z = \frac{\rho g}{24IE}(x^4 + C_1x^3 + C_2x^2 + C_3x + C_4)$. Boundary conditions: clamped on 2 sides $z(0) = z(L) = z'(0) = z'(L) = 0$, supported on 2 sides $z(0) = z(L) = z''(0) = z''(L) = 0$. For clamped on 1 side use $EIz^{(4)} = -f\delta(x - L)$, $z^{(3)} = -\frac{f}{EI}$, where f is the force acting on the end, and BC $z(0) = z'(0) = z''(L) = 0$.

2.8.2 The Energy of a Deformed Rod

Coordinate system ξ, η, ζ, where ζ is parallel to axis of rod.

Relative rotations are described by the vector $\mathrm{d}\boldsymbol{\varphi}$. Deformation is determined by $\frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}l}$.

Energy can be written as

$$F = \int \frac{1}{2}I_1E\left(\frac{\mathrm{d}\varphi_\xi}{\mathrm{d}l}\right)^2 + \frac{1}{2}I_2E\left(\frac{\mathrm{d}\varphi_\eta}{\mathrm{d}l}\right)^2 + \frac{1}{2}C\left(\frac{\mathrm{d}\varphi_\zeta}{\mathrm{d}l}\right)^2\mathrm{d}l$$

where the first two terms correspond to the previously derived elastic energy and the third term corresponds to the energy stored in twisting/torsion.

Derivation To obtain the bending elastic energy use $(\varphi_\xi, \varphi_\eta) = \boldsymbol{\tau} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}l} \approx \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}x}$, then $\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}l} \approx (\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}, \frac{\mathrm{d}^2 y}{\mathrm{d}x^2})$.

Torsion for cylinder has deformations $u_{xz} = -\frac{y}{2}\frac{\mathrm{d}\varphi}{\mathrm{d}z}$, $u_{yz} = \frac{x}{2}\frac{\mathrm{d}\varphi}{\mathrm{d}z}$. Energy and torque needed to twist the top an angle φ_0 (C is the torsional rigidity)

$$F = \frac{C}{2}\int\left(\frac{\mathrm{d}\varphi}{\mathrm{d}z}\right)^2\mathrm{d}z,\ C = \frac{\pi}{2}\mu R^4,$$
$$M = C\frac{\mathrm{d}\varphi}{\mathrm{d}z} = \frac{\pi}{2}\frac{\mu\varphi_0 R^4}{l}.$$

Derivation Torsion by angle φ has $u_x = -y\varphi(z)$, $u_y = x\varphi(z)$, $\operatorname{div}\boldsymbol{u} = 0$. This gives u_{xz}, u_{yz}, other $u_{ik} = 0$. Stress $\sigma_{ik} = 2\mu u_{ik}$, then $F = \int \mu u_{ik}^2\,\mathrm{d}z\,\mathrm{d}^2r = ... = \frac{\pi R^4}{2}\mu\int\frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}z}\right)^2\mathrm{d}z$. For torque, add energy due to external force V and vary F with respect to φ. Use $\delta V = -M\delta\varphi$ and integration by parts for integral. $\delta\varphi$ is arbitrary, integral and bracket need to vanish independently.

2.8.3 Deformation of an elastic Medium when a Point Force is applied

Equation to solve $\nabla^2\boldsymbol{u} + \frac{1}{1-2\sigma}\operatorname{grad}\operatorname{div}\boldsymbol{u} = -2\frac{1+\sigma}{E}\boldsymbol{F}\delta(\boldsymbol{r})$.
Deformation in 1D and 3D

$$\boldsymbol{u} = \frac{1+\sigma}{8\pi E(1-\sigma)}\frac{(3-4\sigma)\boldsymbol{F} + \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{F})}{r} \qquad (3D)$$
$$\boldsymbol{u} = \frac{F}{2C}|z| \qquad (1D)$$

Derivation *In 3D*: Solve by switching to Fourier space $k^2\boldsymbol{u} + \frac{1}{1-2\sigma}\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{u}) = 2\frac{1+\sigma}{E}\boldsymbol{F}$. Multiply by $\boldsymbol{k}$, extract $\boldsymbol{k}\cdot\boldsymbol{u} = ...$ and insert back to original Fourier equation to obtain expression for $\boldsymbol{u}(k) \propto \boldsymbol{F}/k^2 - \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{F})/k^4$. Transform back using $1/k^2 \to \frac{1}{4\pi r}$, $\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{F}) \to -\nabla(\boldsymbol{F}\cdot\nabla)f(\boldsymbol{r})$, $\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{F})/k^4 \to \frac{\pi^2}{(2\pi)^3}\nabla(\boldsymbol{F}\cdot\nabla)r = \frac{1}{8\pi}\frac{\boldsymbol{F}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{F})}{r}$ with $\boldsymbol{n} = \boldsymbol{r}/r$.
In 1D: Vary $F = \int\left(\frac{\mathrm{d}u}{\mathrm{d}z}\right)^2 + F\delta(z)u\,\mathrm{d}z$ to obtain $u = \alpha|z|$. Integrate this solution as $C\frac{\mathrm{d}u}{\mathrm{d}z}|^{+0} + -0 = F$ to get α.

2.8.4 Point Force applied to Surface

Equation to solve $\mu\nabla^2\boldsymbol{u} + (\mu+\lambda)\operatorname{grad}\operatorname{div}\boldsymbol{u} = 0$ in cylindrical coordinates with BC $\sigma_{rz}(z=0) = \sigma_{\varphi z}(z=0) = 0, \sigma_{zz} - P\delta^2(\boldsymbol{r})$.

$$u_z = -\frac{\alpha}{2R}\left(\frac{2\mu+\lambda}{\mu} + \frac{\mu+\lambda}{\mu}\frac{z^2}{R^2}\right)$$
$$u_r = \frac{\alpha}{2r}\left(1 - \frac{2\mu+\lambda}{\mu}\frac{z}{R} + \frac{\mu+\lambda}{\mu}\frac{z^3}{R^3}\right)$$
$$\sigma_{zz} = 3\alpha(\mu+\lambda)\frac{z^3}{R^5}$$
$$\sigma_{\varphi\varphi} = \frac{\alpha\mu}{r^2}\left(1 - \frac{2z}{R} + \frac{z^3}{R^3}\right)$$

Neutral angle for u_r: $\sin\beta = \frac{z}{R} = \sqrt{\frac{1}{4} + \frac{\mu}{\mu+\lambda}} - \frac{1}{2}$
Neutral angle for $\sigma_{\varphi\varphi}$: $\sin\beta' = \frac{\sqrt{5}-1}{2} \approx 38.2°$
Derivation Take div of equation to get $\nabla^2\operatorname{div}\boldsymbol{u} = 0$. Use Ansatz $\operatorname{div}\boldsymbol{u} = -\alpha\frac{\partial}{\partial z}\frac{1}{R} = \alpha\frac{z}{R^3}$ and $u_z = \alpha\frac{\mu+\lambda}{\mu}\frac{\partial^2 R}{\partial z^2}$ with $R = \sqrt{r^2+z^2}$ to solve initial equation in z component for u_z by using $1/R = \nabla^2 R/2$ to eliminate ∇^2. Add harmonic function to get $u_z = \frac{\gamma}{R} - \alpha\frac{\mu+\lambda}{2\mu}\frac{z^2}{R^3}$. Use $\operatorname{div}\boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}$ to solve for u_r: $\frac{\partial}{\partial r}(ru_r) = r\left(\operatorname{div}\boldsymbol{u} - \frac{\partial u_z}{\partial z}\right)$. Use $ru_r = 0$ at $r = 0$ as BC. BC at $z = 0$ states $\sigma_{rz} = 2\mu u_{rz} = 0$. Use $R \approx r\left(1 + \frac{z^2}{2r^2}\right)$ to obtain $\gamma = -\alpha\frac{2\mu+\lambda}{2\mu}$. Calculate $\int\sigma_{zz}\,\mathrm{d}^2r = \int\frac{z^3}{(z^2+r^2)^{5/2}}\,\mathrm{d}^2r = \frac{2\pi}{3}$ and hence at surface $\sigma_{zz}(z=0) = -P\delta^2(\boldsymbol{r})$ we get $\alpha = -\frac{P}{2\pi(\mu+\lambda)}$. For the neutral angle solve $u_r/\ \sigma_{zz}$ for $z/R = \sin\beta$.

Interaction energy of two balls displacing the surface $U_{int} = F(\boldsymbol{u}_1 + \boldsymbol{u}_2) = -\int P_j\delta^2(\boldsymbol{r} - \boldsymbol{r}_2)u_{1j}\,\mathrm{d}\boldsymbol{S} = -Pu_{1z}(\boldsymbol{r}_2) = -\frac{P^2}{4\pi}\frac{2\mu+\lambda}{\mu(\mu+\lambda)}\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}$

Derivation Define $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$. Compute $\lambda(\operatorname{div}\boldsymbol{u})^2 + \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)^2$ for $\boldsymbol{u}_1, \boldsymbol{u}_2$. Extract $F(\boldsymbol{u}_1), F(\boldsymbol{u}_2)$, keep mixed terms and use partial integration on their integral such that one volume integral and two surface integral remain $\int(-\lambda\cdot ... - \mu\cdot ...)u_{2j}\,\mathrm{dV} + \int(\lambda\cdot ... + \mu\cdot ... - Pj\delta^2(\boldsymbol{r}-\boldsymbol{r}_1))u_{2j}\,\mathrm{d}S - \int P_j\delta^2(\boldsymbol{r}-\boldsymbol{r}_2)u_{1j}\,\mathrm{d}S$. The first integral vanishes because it is the equation equilibrium in the bulk, the second integral vanished because it is the boundary conditions at the surface.

3 Elastic Waves

3.1 Wave Equation

$E = E_{el} + E_{kin} = \int\frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x + \int\frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2\mathrm{d}x$

Wave equation $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ with $c = \sqrt{T/\rho}$ and general solution $u = f(x - ct) + g(x + ct)$.

Derivation Vary total energy or use Newton's second law $\boldsymbol{F} = m\boldsymbol{a}$.

3.2 Elastic Waves in isotropic medium

Eq of motion $\rho \ddot{\boldsymbol{u}} = \mu \nabla^2 \boldsymbol{u} + (\mu + \lambda)\,\text{grad div}\,\boldsymbol{u}$

Longitudinal waves satisfy $\text{rot}\,\boldsymbol{u}_l = 0$, $\rho \ddot{\boldsymbol{u}}_l = (2\mu + \lambda)\nabla^2 \boldsymbol{u}_l$ and $c_l = \sqrt{\frac{2\mu+\lambda}{\rho}} \sim \sqrt{\frac{K}{\rho}}$.

Transverse waves satisfy $\text{div}\,\boldsymbol{u}_t = 0$, $\rho \ddot{\boldsymbol{u}}_t = \mu \nabla^2 \boldsymbol{u}_t$ and $c_t = \sqrt{\frac{\mu}{\rho}}$.

Derivation Use $\boldsymbol{u} = \boldsymbol{u}_l + \boldsymbol{u}_t$. For longitudinal $\nabla^2 \boldsymbol{u}_l = \text{grad div}\,\boldsymbol{u}_l - \text{rot rot}\,\boldsymbol{u}_l = \text{grad div}\,\boldsymbol{u}_l$. Speed of wave can be obtained by comparing coefficients.

Monochromatic plane waves $\boldsymbol{u} = \Re\left(\boldsymbol{A}_k e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}\right)$ have for longitudinal waves $\boldsymbol{A}_k \parallel \boldsymbol{k}$, dispersion $w_l = c_l k$ while for transverse waves $\boldsymbol{A}_k \perp \boldsymbol{k}$, dispersion $w_t = c_t k$.

Polarization for transverse waves $\boldsymbol{u} = \boldsymbol{A}_1 \cos\omega t + \boldsymbol{A}_2 \sin\omega t$. Linear polarization for $\boldsymbol{A}_1 \parallel \boldsymbol{A}_2$, circular polarization $\boldsymbol{A}_1 \perp \boldsymbol{A}_2$, $|\boldsymbol{A}_1| = |\boldsymbol{A}_2|$.

3.3 Elastic Waves in Crystals

Equation of motion $\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ik}}{\partial x_k} = \lambda_{iklm} \frac{\partial^2 u_m}{\partial x_k \partial x_l}$

Dispersion relation for Ansatz $\boldsymbol{u}(\boldsymbol{r}, t) = \boldsymbol{A} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$ yields condition $\lambda_{iklm} \boldsymbol{k}_k \boldsymbol{k}_l = \rho\omega^2 \delta_{im}$.

Derivation Use $\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ik}}{\partial x_k}$ with $\sigma_{ik} = \lambda_{iklm} u_{lm}$. Pluck Ansatz into equation to obtain dispersion relation.

Example: cubic crystal with $\lambda_{xxxx} = C_{11}, \lambda_{xxyy} = C_{12}, \lambda_{xyxy} = \lambda_{xzxz} = C_{44}$. For $\boldsymbol{k} = (k, 0, 0)$ we get $w_l^2 = \frac{C_{11}}{\rho} k^2$, $w_t^2 = \frac{C_{44}}{\rho} k^2$.

3.4 Reflection at free Surface

Reflection mixes waves Purely longitudinal or transverse waves are mixed at reflection. It must hold that $\omega = \omega'$ due to continuity, $k_\parallel = k'_\parallel$ due to y-symmetry, hence $k \sin\theta = k \sin\theta'$. Since $k = \frac{\omega}{c}, k' = \frac{\omega}{c'}$ we get $\frac{\sin\theta}{\sin\theta'} = \frac{c}{c'} = n$.

Reflection for waves of the form $\boldsymbol{u} = A_0 \boldsymbol{n}_0 e^{i\boldsymbol{k}_0\cdot\boldsymbol{r}} + A_l \boldsymbol{n}_l e^{i\boldsymbol{k}_l\cdot\boldsymbol{r}} + A_t(\hat{\boldsymbol{z}} \times \boldsymbol{n}_t) e^{i\boldsymbol{k}_t\cdot\boldsymbol{r}}$ we get with $n = \frac{c_l}{c_t}$

$$A_l = A_0 \frac{\sin^2\theta_t \sin 2\theta_0 - n^2 \cos^2 2\theta_t}{\sin 2\theta_t \sin 2\theta_0 + n^2 \cos^2 2\theta_t},$$
$$A_t = -A_0 \frac{2n \sin 2\theta_0 \cos 2\theta_t}{\sin 2\theta_t \sin 2\theta_0 + n^2 \cos^2 2\theta_t}.$$

Derivation Note that $n_{0,x} = n_{l,x} = \cos\theta_0$, $n_{0,y} = -n_{l,y} = \sin\theta_0$ and $\hat{\boldsymbol{z}} \times \boldsymbol{n}_t = (sin\theta_t, \cos\theta_t)$. Use the Ansatz to derive u_{xx}, u_{xy} (note how u_{ll} would look like). From BC $\sigma_{xx} = \sigma_{yx} = 0$ and Hooke's law $\sigma_{ik} = 2\rho c_t^2 u_{ik} + \rho(c_l^2 - 2c_t^2) u_{ll} \delta_{ik}$. Equations for A_0, A_l, A_t. For $\theta_0 = 0$, $A_l = -A_0$, $A_t = 0$, longitudinal reflected wave.

3.5 Surface Waves

Ansatz $u \propto e^{i(kx - \omega t) + \chi z}$ with $\chi = \sqrt{k^2 - \frac{\omega^2}{c^2}}$ and boundary condition $\sigma_{ik} n_k = 0$.

Dispersion relation for reflected surface waves $\omega = c_t k \xi$ with $\xi < 1$ the solution of $(1 - \frac{1}{2}\xi^2)^4 = (1 - \xi^2)(1 - \frac{c_t^2}{c_l^2}\xi^2)$ that is within the range $c_{\text{surface}} = c_t \xi < c_t < c_l$.

Derivation Pluck Ansatz into equation to obtain χ. $\sigma_{iz} = 0$, because $\boldsymbol{n} \parallel \boldsymbol{z}$. Then $u_{iz} = 0$ and $\sigma(u_{xx} + u_{yy}) + (1 - \sigma)u_{zz} = 0$. Because of this and the Ansatz $u_y = 0$. Wave parts satisfy $\frac{\partial u_{tx}}{\partial x} + \frac{\partial u_{tx}}{\partial z} = 0$, $\frac{\partial u_{lx}}{\partial z} - \frac{\partial u_{lz}}{\partial x} = 0$. Use Ansatz (with constants a, b for transverse/longitudinal respectively) to derive $u_{tx}, u_{tz}, u_{lx}, u_{lz}$.

- *BC1*: $0 = \sigma_{xz} \propto u_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$. Substitute in $u_{i/l,x/z}$ to obtain $a(\chi_t^2 + k^2) + 2bk\chi_l = 0$.
- *BC2*: $0 = \sigma_{zz} = c_l^2 \frac{\partial u_z}{\partial z} + \left(c_l^2 - 2c_t^2\right) \frac{\partial u_x}{\partial x}$. Use $\boldsymbol{u} = \boldsymbol{u}_t + \boldsymbol{u}_l$, $\frac{\partial u_{tz}}{\partial z} + \frac{\partial u_{tx}}{\partial x} = 0$, $\omega^2 = c_{t,l}^2(k^2 - \chi_{t,l}^2)$ to obtain $2a\chi_t k + b(k^2 + \chi_t^2) = 0$.

BC1,2 compatible if $(k^2 + \chi_t^2)^2 = 4k^2 \chi_l \chi_t$. Use $\chi_{l,t}^2 = k^2 - \omega^2/c_{l,t}^2$ to get $(2k^2 - \omega^2/c_t^2)^4 = 16k^4(k^2 - \omega^2/c_t^2)(k^2 - \omega^2/c_l^2)$. Note that $\omega \propto k$, hence Ansatz $\omega = c_t k \xi$. Pluck in and solve cubic equation in $x = \xi^2$. It follows the dispersion relation and $c_{\text{surface}} = c_t \xi < c_t < c_l$.

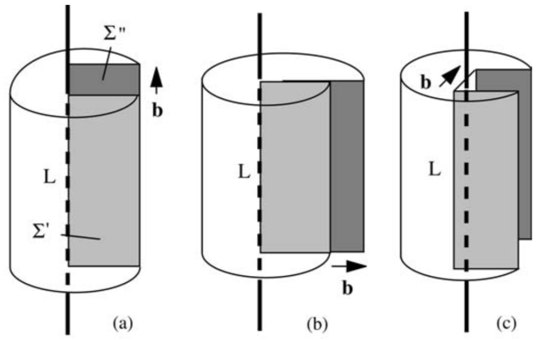
4 Dislocations

4.1 Stress Estimation

$\sigma = \frac{\mu}{2\pi}\sin\frac{2\pi u}{a}$ gives maximal stress $\sigma_{max} = \frac{\mu}{2\pi} \sim \frac{\mu}{10}$. However, due to dislocations in reality $\sigma \sim 10^{-4}\mu$.

Derivation Consider periodic crystal with distance a. For small u strain is $\frac{u}{a}$, stress $\mu\frac{u}{a}$. Assume periodic function $\sigma \propto \sin\frac{2\pi u}{a}$, because upon displacement of $\sim a$ the lattice retains original form. For $u \ll a$ $\sigma \sim \mu\frac{u}{a}$ hence $\sigma = \frac{\mu}{2\pi}\sin\frac{2\pi u}{a}$. Maximal stress $\sigma_{max} = \frac{\mu}{2\pi} \sim \frac{\mu}{10}$.

4.2 Definitions & Displacement Field



Along dislocation, $\boldsymbol{u}$ is a multivalued function, derivative, however, are single-valued. The figure shows one screw dislocation (a), and two edge dislocations (b,c).

Distortion tensor $w_{ik} = \frac{\partial u_k}{\partial x_i}$, $u_{ik} = \frac{1}{2}(w_{ik} + w_{ki})$.

Burger's vector $\boldsymbol{b}_i = -\oint \mathrm{d}\boldsymbol{u}_i = -\oint \frac{\partial u_i}{\partial x_k}\,\mathrm{d}x_k = -\oint_L w_{ik}\,\mathrm{d}x_k$. It is independent of path. Dislocations cannot end inside the sample.

Tau $\boldsymbol{\tau}$ is the tangent vector at the given point of the dislocation. It is along the direction of elongation of the dislocation. The dislocation line is a curve along which the angle between $\boldsymbol{b}, \boldsymbol{\tau}$ is changing.

Screw dislocations $\boldsymbol{b} \parallel \boldsymbol{\tau}$

Edge dislocations $\boldsymbol{b} \perp \boldsymbol{\tau}$

Equation of equilibrium containing dislocations

$$\frac{\partial w_{ki}}{\partial x_k} + \frac{1}{1-2\sigma}\frac{\partial w_{ll}}{\partial x_i} = [\boldsymbol{\tau} \times \boldsymbol{b}]_i \delta^2(\xi)$$

$$\Delta\boldsymbol{u} + \frac{1}{1-2\sigma}\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} = [\boldsymbol{\tau} \times \boldsymbol{b}]\delta^2(\xi)$$

Derivation $-b_k = \oint_L w_{ik}\,\mathrm{d}x_i = \int_{S_L} e_{ilm}\frac{\partial w_{mk}}{\partial x_l}\,\mathrm{d}\boldsymbol{S}_i$. Because e_{ilm} antisymmetric, $\frac{\partial w_{mk}}{\partial x_l}$ symmetric, $e_{ilm}\frac{\partial w_{mk}}{\partial x_l} = 0$ everywhere apart from the crossing point of dislocation line with surface S_L $\implies$ $e_{ilm}\frac{\partial w_{mk}}{\partial x_l} = -\tau_i \boldsymbol{b}_k \delta^2(\xi)$ or $\frac{\partial w_{nk}}{\partial x_k} - \frac{\partial w_{kk}}{\partial x_n} = -[\boldsymbol{\tau} \times \boldsymbol{b}]_n \delta^2(\xi)$. Rewrite equation of equilibrium with w_{ik} and insert.

4.3 Screw Dislocation

Deformation $u_z = \frac{b}{2\pi}\varphi$

Derivation $\boldsymbol{u}(x,y) \parallel z \implies \mathrm{div}\,\boldsymbol{u} = 0 \implies \Delta u_z = 0 \implies u_z = \frac{b}{2\pi}\varphi$.

Energy of screw dislocation $E = \frac{\mu b^2}{4\pi}\log\frac{R}{b}$, where R is either the system size or the size of the dislocation.

Derivation $u_{z\varphi} = \frac{b}{4\pi r}, \sigma_{z\varphi} = 2\mu u_{z\varphi}$ and other components zero. $E = \frac{1}{2}\int \sigma_{ik}u_{ik}\,\mathrm{d}^2 r$.

4.4 Edge Dislocation

Deformation $u_x = \frac{b}{2\pi}\left(\arctan\frac{y}{x} + \frac{1}{2(1-\sigma)}\frac{xy}{x^2+y^2}\right)$ $u_y = -\frac{b}{2\pi}\left(\frac{1-2\sigma}{2(1-\sigma)}\log\sqrt{x^2+y^2} + \frac{1}{2(1-\sigma)}\frac{x^2}{x^2+y^2}\right)$

Stress $\sigma_{xx} = -bB\frac{y(3x^2+y^2)}{(x^2+y^2)^2}$, $\sigma_{yy} = bB\frac{y(x^2-y^2)}{(x^2+y^2)^2}$, $\sigma_{xy} = bB\frac{x(x^2-y^2)}{x^2+y^2}$.

Energy of edge dislocations $E = \frac{\mu b^2}{4pi(1-\sigma)}\log\frac{R}{b}$ and $F = \frac{1}{2}b\int_0^R \sigma_{xy}(\varphi=0)\,\mathrm{d}x$.

Derivation Equation to solve $\nabla^2\boldsymbol{u} + \frac{1}{1-2\sigma}\nabla\,\mathrm{div}\,\boldsymbol{u} = b\boldsymbol{e}_y\delta^2(\boldsymbol{r})$. Look for solution of the form $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{w}$ with $u_{0,x} = \frac{b}{2\pi}\varphi, u_{0,y} = \frac{b}{2\pi}\log r$ taking care of the multivaluedness. Since $\mathrm{div}\,\boldsymbol{u}_0 = 0$, $\Delta\boldsymbol{u}_0 = b\boldsymbol{e}_y\delta^2(\boldsymbol{r})$, $\boldsymbol{w}$ is single-valued and satisfies same equation to solve. Solve by switching to Fourier space with solution $w = \frac{b}{4\pi(1-\sigma)}\int \frac{3-4\sigma}{R}\boldsymbol{e}_y + \frac{y}{R^3}\boldsymbol{r}\,\mathrm{d}z'$, $R = \sqrt{r^2+z'^2}$. Derive u_x, u_y from $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{w}$. Derive $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$. Energy is $E = \frac{\mu b^2}{4\pi^2(1-\sigma)}\int \frac{y^2}{r^4}\,\mathrm{d}^2 r$.

Cut Surface S_D Define $\boldsymbol{u}$ as continuous function on plane with cut surface S_D sucht that $\boldsymbol{u}_+ - \boldsymbol{u}_-|_{S_D} = \boldsymbol{b}$. Then $F = \frac{1}{2}\int_R \sigma_{ij}u_{ij}\,\mathrm{d}^2 r = \frac{1}{2}b\int_0^R \sigma_{xy}(\varphi=0)\,\mathrm{d}x$.

4.5 Dislocation Motion

S_D-surface is the surface where displacement jumps $\boldsymbol{u}_+ - \boldsymbol{u}_-|_{SD} = \boldsymbol{b}$.

Change of volume $\delta V = \boldsymbol{b}\delta\boldsymbol{S} = \delta\boldsymbol{x}\cdot[\boldsymbol{\tau}\times\boldsymbol{b}]\,\mathrm{d}l$. I.e. screw dislocations never change the volume.

Glide motion is parallel to $\boldsymbol{\tau}, b$, does not change the volume $\delta V = 0$ and hence is easy motion.

Climb motion does change the volume $\delta V \neq 0$ and hence is hard to achieve. For it to happen, atoms have to diffuse.

4.6 Forces acting of Dislocations

Plastic deformation on moving dislocation by $\delta \boldsymbol{r}$ $\delta u_{ik}^{(pl)} = \frac{1}{2}\left(b_i[\delta \boldsymbol{r} \times \boldsymbol{\tau}]_k + b_k[\delta \boldsymbol{r} \times \tau]_i\right)\delta^2(r - r_d)$.

Derivation On surface S_D: $\boldsymbol{u}_+ - \boldsymbol{u}_- = \boldsymbol{b}$, thus w_{ik} has singularity there $w_{ik}^{(S)} = n_i b_k \delta(\xi)$, where $\boldsymbol{n}$ is normal to surface $\boldsymbol{\xi} \parallel \boldsymbol{n}$. Dislocation motion is changing S_D, then by moving dislocation by $\delta \boldsymbol{r}$ we obtain the above equation for plastic deformation.

Peach Köhler force $f_i = e_{ikl}\tau_k \sigma_{lm} b_m$

Derivation Work due to external sources $\delta R = \int \sigma_{ik}^{ext} \delta u_{ik}\, \mathrm{dV} = \oint \sigma_{ik}^{ext} e_{ilm} \delta r_l \tau_m b_k \, \mathrm{d}l = \oint f_i \delta r_l \, \mathrm{d}l$ by substituting $\delta u_{ik}^{(pl)}$. Force by comparing coefficients.

Interaction of two dislocations has the forces $f_x = b_1 b_2 B \frac{x(x^2 - y^2)}{r^4}$, $f_y = b_1 b_2 B \frac{y(3x^2 + y^2)}{r^4}$. Aligned along the same direction $b_1 b_2 > 0$, there is an unstable eq point at $x = y$. Aligned the opposite direction $b_1 b_2 < 0$, the opposite case holds.

Derivation Use coordinate system such that $\tau_z = -1$, $b_x = b$ and pluck into Peach Köhler force. Use expressions for σ_{ij} from before. Point is in equilibrium in x-direction for $x^2 = y^2$ (unstable) and $x = 0$ (stable). However, $|f_y|$ always increases.

4.7 Peierls-Nabarro Force

Peierls-Nabarro force $F = \frac{2\pi\mu b}{1-\sigma} \sin \frac{2\pi x}{b} e^{-\frac{2\pi |y_0|}{b}}$

Critical stress $\sigma_{max} = \mu e^{-\pi}$

Derivation $x_n = nb, y_m = mb + \frac{b}{2}$. Start with $E = \frac{\mu b^2}{4\pi^2(1-\sigma)} b^2 \sum_{n,m} \frac{y_m^2}{((x-x_n)^2 + y_m^2)^2}$. Rewrite $E = -\frac{\mu b^4}{4\pi^2(1-\sigma)} \sum_m y_m^2 \frac{\partial}{\partial y_m^2} \sum_n \frac{1}{(x-nb)^2 + y_m^2}$. Calculate last sum using Poisson formula $\sum_n \frac{1}{(x-nb)^2 + y_m^2} = \frac{\pi}{b|y_m|} \sum_k \exp\left(i \frac{2\pi k x}{b}\right) \exp\left(-\frac{2\pi |k y_m|}{b}\right)$. Keep only largest terms with $k = \pm 1$ and smallest y_m to obtain $E \approx \frac{\mu b^2}{1-\sigma} \cos \frac{2\pi x}{b} \exp\left(-\frac{\pi |y_0|}{b}\right)$. Use $y_0 = b/2$ for σ_{max}. Calculate force $F = \frac{\mathrm{d}E}{\mathrm{d}x}$.

5 Hydrodynamics: Basic Equations

We need three quantities, the fluid velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ and two thermodynamic quantities, e.g. the pressure $p(\boldsymbol{r}, t)$ and the density $\rho(\boldsymbol{r}, t)$.

5.1 Continuity Equation

Continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \boldsymbol{v} = 0 \qquad \text{(Continuity)}$$

Derivation Change of mass $\frac{\partial}{\partial t} \int \rho \, \mathrm{dV}$ is the flow out of the surface $-\oint \rho \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}$. Then Gauss' theorem.

5.2 Euler's Equation

Ideal Fluid A fluid without viscosity and thermal conductivity is called **ideal**.

Euler's equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} + \boldsymbol{f} \qquad \text{(Euler's)}$$
$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}_i) = -\frac{\partial \Pi_{ik}}{\partial x_k} + \rho \boldsymbol{f}$$

Assumptions We neglect energy dissipation, internal friction (viscosity) and heat exchange.

Derivation $\boldsymbol{F} = -\oint p \, \mathrm{d}\boldsymbol{S} = \int \rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \mathrm{dV}$ and $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial v}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$

5.3 Hydrostatics & Convection

Hydrostatic equations $\alpha \sim 6.5°/km$

$$p = p(0) - \rho g z$$
$$p = p(0) \exp\left(-\frac{mgz}{T}\right) \qquad \text{(Boltzmann's law)}$$
$$p = p(0) \exp\left(1 - \frac{\alpha z}{T_0}\right)^{mg/\alpha}$$

Derivation Fluid at rest: $\operatorname{grad} p = \rho \boldsymbol{g}$ (Euler's equation). For first equation (incompressible fluid) direct integration, for second use ideal gas law $\rho = \frac{pm}{T}$, for third use linear temperature decay $T(z) = T_0 - \alpha z$.

Why wind blows and current flows p, ρ determine temperature. Because $\frac{\partial p}{\partial z} = \rho \boldsymbol{g}$ p, ρ and T should be functions of altitude z only.

Convection

$$-\frac{\mathrm{d}T}{\mathrm{d}z} < \frac{g\beta T}{C_p} \approx 10\,\mathrm{K\,km^{-1}}, \quad \beta = \frac{1}{V}\left.\frac{\partial V}{\partial T}\right|_p$$

Assumptions Substance expands on heating, shift is adiabatic.

Derivation Adiabatically up-shifted fluid element needs to be forced back down, i.e. must be heavier than displaced fluid $V(S(z-\mathrm{d}z), z)|_p < V(S(z), z)|_p$ or $\left(\frac{\partial V}{\partial S}\right)_p \frac{\mathrm{d}S}{\mathrm{d}z} > 0$. Then $0 < \frac{\mathrm{d}S}{\mathrm{d}z} = \left(\frac{\partial S}{\partial T}\right)_p \frac{\mathrm{d}T}{\mathrm{d}z} + \left(\frac{\partial S}{\partial p}\right)_T \frac{\mathrm{d}p}{\mathrm{d}z} = \frac{c_p}{T}\frac{\mathrm{d}T}{\mathrm{d}z} + \left(\frac{\partial V}{\partial T}\right)_p \frac{g}{V}$, where $V = \frac{1}{\rho}$, $\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g$, $\frac{\partial S}{\partial T} = \frac{c_p}{T}$, because $\left(\frac{\partial V}{\partial S}\right)_p = \frac{T}{c_p}\left(\frac{\partial V}{\partial T}\right)_p$.

5.4 Bernoulli's Equations

Streamlines are lines such that $\frac{\mathrm{d}x}{\boldsymbol{v}_x} = \frac{\mathrm{d}y}{\boldsymbol{v}_y} = \frac{\mathrm{d}z}{\boldsymbol{v}_z}$.

Bernoulli's Along streamlines it holds that

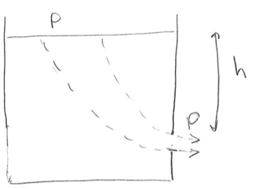
$$H + \frac{1}{2}\boldsymbol{v}^2 = const. \qquad \text{(Bernoulli's)}$$

$$\rho g z + p + \frac{1}{2}\rho \boldsymbol{v}^2 = const.$$

Assumptions Isentropic motion, steady flow.

Derivation Define enthalpy per unit mass $H = U + pV$. For isentropic motion ($\mathrm{d}S = 0$) it holds that $\mathrm{d}H = V\,\mathrm{d}p = \mathrm{d}p/\rho$ and Euler's eq becomes $\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla H$. Be rewriting non-linear term $\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\times(\nabla\times\boldsymbol{v}) = -\nabla(H + \frac{1}{2}\boldsymbol{v}^2)$. Use steady flow. Multiplying with unit vector along streamlines $\boldsymbol{l}$ renders the left side 0, hence $\frac{\partial}{\partial l}(H + \frac{1}{2}\boldsymbol{v}^2) = 0$.

Torricelli law $|\boldsymbol{v}| = \sqrt{2gh}$



Assumptions $v(0) = 0$, $p(0) = p(-h)$, uncompressible liquid.

Derivation Use $gz + \frac{p}{\rho} + \frac{1}{2}\boldsymbol{v}^2 = const.$ (Bernoulli) at $z = -h$ and $z = 0$ with $v(0) = 0$.

5.5 Energy and Momentum Flux

Energy Flux Density

$$\frac{\partial}{\partial t}\left(\rho E + \frac{1}{2}\rho\boldsymbol{v}^2\right) = \rho\boldsymbol{v}\left(h + \frac{1}{2}\boldsymbol{v}^2\right)$$
(Energy Flux Density)

Derivation We need to compute $\frac{\partial}{\partial t}\left(\rho E + \frac{1}{2}\rho\boldsymbol{v}^2\right)$.

- $\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\boldsymbol{v}^2\right)$: Use continuity and Euler's equations and $\boldsymbol{v}\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \boldsymbol{v}\cdot\frac{\nabla\boldsymbol{v}^2}{2}$. Rewrite $\mathrm{d}h = T\,\mathrm{d}s + V\,\mathrm{d}p$ to $\nabla p = \rho\nabla h - \rho T\nabla s$. Final result $\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\boldsymbol{v}^2\right) = -\frac{\boldsymbol{v}^2}{2}\operatorname{div}(\rho\boldsymbol{v}) - \rho\boldsymbol{v}\cdot\nabla\left(h + \frac{1}{2}\boldsymbol{v}^2\right) + \rho T(\boldsymbol{v}\cdot\nabla)s$.
- $\frac{\partial\rho E}{\partial t}$: Use $\mathrm{d}E = T\,\mathrm{d}s - p\,\mathrm{d}V = T\,\mathrm{d}s + \frac{p\,\mathrm{d}\rho}{\rho^2}$, rewrite $\mathrm{d}(\rho E) = E\,\mathrm{d}\rho + \rho\,\mathrm{d}E = h\,\mathrm{d}\rho + \rho T\,\mathrm{d}s$, then $\frac{\partial(\rho E)}{\partial t} = h\frac{\partial\rho}{\partial t} + \rho T\frac{\partial s}{\partial t}$. Use continuity eq, $\frac{\partial S}{\partial t} = \frac{\mathrm{d}S}{\mathrm{d}t} - (\boldsymbol{v}\cdot\nabla)S$ and adiabaticity $\frac{\mathrm{d}S}{\mathrm{d}t} = 0$. Final result $\frac{\partial(\rho E)}{\partial t} = -H\operatorname{div}(\rho\boldsymbol{v}) - \rho T(\boldsymbol{v}\cdot\nabla)S$.

Combine $\frac{\partial}{\partial t}\left(\rho E + \frac{1}{2}\rho\boldsymbol{v}^2\right) = -\operatorname{div}(\rho\boldsymbol{v}(H + \frac{1}{2}\boldsymbol{v}^2))$. Obtain flux from comparing coefficient in $\frac{\partial}{\partial t}\int(\frac{1}{2}\rho\boldsymbol{v}^2 + \rho E)\,\mathrm{dV} = -\oint\rho(H + \frac{1}{2}\boldsymbol{v}^2)\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}$.

Momentum Flux Density Tensor

$$\Pi_{ik} = p\delta_{ik} + \rho\boldsymbol{v}_i\boldsymbol{v}_k \qquad \text{(Momentum Flux)}$$

Derivation Use continuity and Eulers to calculate $\frac{\partial(\rho\boldsymbol{v}_i)}{\partial t} = \rho\frac{\partial v_i}{\partial t} + \frac{\partial\rho}{\partial t}v_i = -\rho v_k\frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - v_i\frac{\partial(\rho v_k)}{\partial x_k} = -\frac{\partial p}{\partial x_i} - \frac{\partial(\rho v_i v_k)}{\partial x_k}$. Obtain Π_{ik} from comparing to $\frac{\partial(\rho v_i)}{\partial t} = -\frac{\partial\Pi_{ik}}{\partial x_k}$.

5.6 Circulation

Vorticity is defined as $\boldsymbol{\Omega} = \operatorname{rot}\boldsymbol{v}$.

Velocity circulation around a contour C is defined as

$$\Gamma = \oint_C \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{l}.$$

Law of conservation of circulation for a small fluid surface $\delta\boldsymbol{S}$ it holds that $\delta\boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{v} = const.$ or

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0, \quad \oint \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{l} = const. \qquad \text{(Kelvin's Theorem)}$$

Derivation $\frac{\mathrm{d}}{\mathrm{d}t}\oint_C \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{l} = \oint_C \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot\mathrm{d}\boldsymbol{l} + \oint_C \boldsymbol{v}\cdot\frac{\mathrm{d}\,\mathrm{d}\boldsymbol{l}}{\mathrm{d}t}$. Use $\mathrm{d}\boldsymbol{l}' = \boldsymbol{r}_2 + \boldsymbol{v}(\boldsymbol{r}_2)\,\mathrm{d}t - \boldsymbol{r}_1 - \boldsymbol{v}(\boldsymbol{r}_1)\,\mathrm{d}t = \mathrm{d}\boldsymbol{l} + \mathrm{d}t(\mathrm{d}\boldsymbol{l}\,\cdot$

$\nabla)\boldsymbol{v}$ why? , and $\boldsymbol{v}\cdot(\mathrm{d}\boldsymbol{l}\cdot\nabla)\boldsymbol{v} = \mathrm{d}\boldsymbol{l}\cdot\frac{\nabla\boldsymbol{v}^2}{2}$ and $\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\operatorname{grad} H$ (Euler's) to render both parts zero (closed contour integral over gradient vanishes). For other representation $\oint \boldsymbol{v}\,\mathrm{d}l = \int \operatorname{rot}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S} = \mathrm{d}\boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{v} = const.$

Distance and vorticity equations

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = (\boldsymbol{r}\cdot\nabla)\boldsymbol{v}, \quad \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} = (\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{v}$$

Derivation Position: from geometric considerations of previous derivation with $\mathrm{d}\boldsymbol{l}$. Vorticity: start by Euler's as in the derivation of Bernoulli's equation $\frac{\partial\boldsymbol{v}}{\partial t} - \boldsymbol{v}\times(\nabla\times\boldsymbol{v}) = -\nabla(H + \frac{1}{2}\boldsymbol{v}^2)$, take rot to obtain $\frac{\partial\boldsymbol{\Omega}}{\partial t} = \operatorname{rot}(\boldsymbol{v}\times\boldsymbol{\Omega})$. Use $\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = \ldots$ rule to get $\operatorname{rot}(\boldsymbol{v}\times\boldsymbol{\Omega}) = (\boldsymbol{\Omega}\times\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{\Omega}$ and hence $(\boldsymbol{\Omega}\times\nabla)\boldsymbol{v} = \frac{\partial\boldsymbol{\Omega}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\Omega} = \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}$.

Vortex lines

$$\operatorname{rot}\boldsymbol{v} = \boldsymbol{\Omega} = const. \implies \boldsymbol{v} = \frac{\boldsymbol{\Omega}\times\boldsymbol{r}}{2}$$
$$\operatorname{rot}\boldsymbol{v} = \boldsymbol{\Omega}_0\delta^2(\boldsymbol{r}),\ \operatorname{div}\boldsymbol{v} = 0 \implies \boldsymbol{v} = \frac{\boldsymbol{\Omega}_0\times\boldsymbol{r}}{2\pi r^2}$$

We use a cutoff (vortex core radius) at distance a.

6 Potential Flow

6.1 Incompressible and irrotational Flows

Pressure and density $\Delta\rho = \frac{\Delta p}{c^2}$, $c = \sqrt{\left(\frac{\partial p}{\partial\rho}\right)_S}$.

Derivation For longitudinal waves $c_l = \sqrt{K/\rho}$. $E = \frac{1}{2}VK(\operatorname{div}u)^2 = \frac{1}{2}VK(\frac{\delta V}{V})^2$ with compression modulus K. From thermodynamics $K = V\left(\frac{\partial^2 E}{\partial V^2}\right)_S = -V\left(\frac{\partial p}{\partial V}\right)_S = \rho\left(\frac{\partial p}{\partial\rho}\right)_S$.

Incompressibility means $\rho = const.$ or $\operatorname{div}\boldsymbol{v} = 0$. It is fulfilled if $v \sim \frac{l}{\tau} \ll c$, where l is the typical length scale of velocity change for time scale τ.

Derivation Continuity eq for constant density becomes $\operatorname{div}\boldsymbol{v} = 0$. Bernoulli $\Delta p \sim \rho\boldsymbol{v}^2$, thus $\Delta\rho = \frac{\Delta p}{c^2} \sim \rho\frac{v^2}{c^2}$, thus $\delta p/p \ll 1$ iff $v \ll c$. In nonsteady flow $\frac{\partial\rho}{\partial t} \sim \frac{\delta\rho}{t} \sim \frac{\delta p}{\tau c^2} \sim \frac{\rho v l}{\tau^2 c^2} \ll \rho\operatorname{div}v \sim \frac{\rho v}{l}$ iff $l/c \ll \tau$.

Potential flow or irrotational flow for $\operatorname{rot}\boldsymbol{v} = 0$. Define the velocity potential $\boldsymbol{v} = \operatorname{grad}\varphi$. Euler's equation becomes (if φ' absorbs the constant of integration)

$$0 = \operatorname{grad}\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}\boldsymbol{v}^2 + H\right) = \frac{\partial\varphi'}{\partial t} + \frac{1}{2}\boldsymbol{v}^2 + H.$$

Small oscillations can often be described by an irrotational flow, i.e. $\operatorname{rot}\boldsymbol{v} = 0$.

Derivation Nonlinear term can be neglected, Euler's eq $\frac{\partial\boldsymbol{v}}{\partial t} = -\nabla H$. Take rot to see $\frac{\partial\boldsymbol{\Omega}}{\partial t} = 0$, so $\operatorname{rot}\boldsymbol{v} = const.$, but since avg is zero $\operatorname{rot}\boldsymbol{v} = 0$.

Bernoulli's equation for steady potential flows becomes $H + \frac{1}{2}\boldsymbol{v}^2 = const.$ everywhere.

Derivation Use Euler's eq in potential flow $0 = \operatorname{grad}\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}\boldsymbol{v}^2 + H\right)$ and steady flow $\frac{\partial\varphi}{\partial t} = const.$

Incompressible potential Flow solves this equivalent system of equations

$$\operatorname{div}\boldsymbol{v} = 0,\ \operatorname{rot}\boldsymbol{v} = 0, \qquad \text{BC} \quad \boldsymbol{v}_n = 0,$$
$$\nabla^2\varphi = 0, \qquad \text{BC} \quad \frac{\partial\varphi}{\partial n} = 0.$$

Solutions for an arbitrary shape has in general $\boldsymbol{A}_i = \alpha_{ik}u_k$, where α_{ik} depends on the body shape.

$$\varphi = -\frac{\boldsymbol{A}\cdot\boldsymbol{n}}{r^2},\ \boldsymbol{n} = \frac{\boldsymbol{r}}{r}$$
$$\boldsymbol{v} = 3\frac{(\boldsymbol{A}\cdot\boldsymbol{n})\cdot\boldsymbol{n} - \boldsymbol{A}}{r^3}$$

Derivation Solve by electrostatic analogy. Solutions of Laplace's eq that vanish at infinity are $1/r$, $\frac{\partial^n}{\partial x^n}(\frac{1}{r})$. Symmetry requires that $\varphi \propto \boldsymbol{u}$. Hence $\varphi = A(\boldsymbol{u}\cdot\nabla(\frac{1}{r})) = -A\frac{\boldsymbol{u}\cdot\boldsymbol{n}}{r^2}$.

Solutions for a sphere At the surface of a sphere with $\boldsymbol{A} = \frac{R^3}{2}\boldsymbol{u}$.

$$\varphi = -\frac{R^3}{2r^2}(\boldsymbol{u}\cdot\boldsymbol{n})$$
$$\boldsymbol{v} = \frac{R^3}{2r^3}(3\boldsymbol{n}(\boldsymbol{u}\cdot\boldsymbol{n}) - \boldsymbol{u})$$
$$p = p_0 - \frac{\rho\boldsymbol{u}^2}{8}(9\cos^2\theta - 5) + \rho R\boldsymbol{n}\cdot\dot{\boldsymbol{u}}$$

Derivation At surface of sphere $\boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{u}\cdot\boldsymbol{n}$. Multiplying the general $\boldsymbol{v}$ from above by $\boldsymbol{n}$ yields $A = \frac{R^3}{2}$. Incompressible $H = p/\rho$, then $p = p_0 - \frac{1}{2}\rho\boldsymbol{v}^2 - \rho\frac{\partial\varphi}{\partial t}$.

Solution moves with sphere $\varphi = \varphi(\boldsymbol{r} - \boldsymbol{u}t, \boldsymbol{u})$, calculate $\frac{\partial \varphi}{\partial t} = -\boldsymbol{u}\,\mathrm{grad}\,\varphi + \frac{\partial \varphi}{\partial \boldsymbol{u}} \cdot \dot{\boldsymbol{u}}$. Use this and $\boldsymbol{v}$ from above to get $p(r = R)$.

Energy and Force for a body of arbitrary shape in a potential flow, $A_i = \alpha_{ik} u_k$,

$$E = \frac{1}{2} m_{ik} u_i u_k \qquad \text{(Energy)}$$

$$m_{ik} = \rho(4\pi\alpha_{ik} - V_0 \delta_{ik}) \qquad \text{(Mass tensor)}$$

$$F_i = -\frac{\mathrm{d}}{\mathrm{d}t}(m_{ik} u_k) = -\frac{\mathrm{d}\boldsymbol{P}_i}{\mathrm{d}t}$$

Derivation (Conceptually) $E = \frac{1}{2}\rho \int \boldsymbol{v}^2 \,\mathrm{dV}$ for a sphere containing the body. Rewrite $\boldsymbol{v}^2$. Use incompressibility, $\boldsymbol{u} = \mathit{const.}$, $\mathrm{div}\,(f\boldsymbol{a}) = \boldsymbol{A}\,\mathrm{grad}\,f + f\,\mathrm{div}\,\boldsymbol{a}$, Gauss' theorem, the explicit solutions for φ and $\boldsymbol{v}$ from above, infinitely large sphere to kill one integral and integral averaging. Force by $\mathrm{d}E = -\boldsymbol{F} \cdot \boldsymbol{u}\,\mathrm{d}t$, comparing to E.

6.2 The Force acting on a Body in Potential Flow

Forces parallel to $\boldsymbol{u}$ are called **drag forces**, perpendicular to $\boldsymbol{u}$ are called **lift foces**.

d'Alembert's Paradox Using previous results, in a potential flow with constant velocity $\boldsymbol{u}$ we get $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = 0$, hence all forces vanish.

Derivation *Alternative 1*: In potential with constant velocity $\boldsymbol{u}$ we get $\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = 0$.
Alternative 2: $\boldsymbol{F}_i = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \frac{\partial}{\partial t}\left(\int p v_i \,\mathrm{dV}\right) = \oint \Pi_{ik}\,\mathrm{d}S_k = -\int_S (p\delta_{ik} + \rho v_i v_k)\,\mathrm{d}S_k = 0$. First term vanishes, because pressure is constant along all directions, second term in the infinite surface limit.
Alternative 3: For F assume $\dot{\boldsymbol{u}} = 0$, then under time reversal pressure must not change (symmetry from Eulers eq). This must equal the situation for a space inverse symmetry, where flow direction and pressure invert. Hence $\boldsymbol{F} = \oint p\,\mathrm{d}\boldsymbol{S} = 0$.

Equation of motion for $\boldsymbol{u}$ in pot flow as reaction to an external force $\boldsymbol{f}$

$$\frac{\mathrm{d}}{\mathrm{d}t}(M\delta_{ik} + m_{ik})u_k = f_i.$$

Eq of motion for $\boldsymbol{v}$ in pot flow when the body moves with velocity $\boldsymbol{u}$

$$(M\delta_{ik} + m_{ik})u_k = (m_{ik} + \rho V_0 \delta_{ik})v_k.$$

Derivation If the body moved as fast as the fluid $\boldsymbol{u} = \boldsymbol{v}$, then $\frac{\mathrm{d}Mu_i}{\mathrm{d}t} = \rho V_0 \dot{v}_i$. If the velocities differ, consider additionally the reaction force $\frac{\mathrm{d}}{\mathrm{d}t}(m)_{ik}(v_k - u_k))$. Integrate equation and set constant to zero.

6.3 Two-dimensional Flow

Definition (Stream function Ψ) defined as

$$\boldsymbol{v}_x = \frac{\partial \Psi}{\partial y}, \quad \boldsymbol{v}_y = -\frac{\partial \Psi}{\partial x}.$$

Stream lines $\mathrm{d}\Psi = \boldsymbol{v}_x\,\mathrm{d}y - \boldsymbol{v}_y\,\mathrm{d}x = 0$

Flux through lines $\int_a^b v_n\,\mathrm{d}\boldsymbol{l} = \Psi(b) - \Psi(a)$

6.4 Potential Flow in 2D

2D Potential Flow satisfies the Cauchy-Riemann conditions for φ and Ψ $\boldsymbol{v}_x = \frac{\partial \varphi}{\partial y} = \frac{\partial \Psi}{\partial y}$, $\boldsymbol{v}_y = \frac{\partial \varphi}{\partial x} = -\frac{\partial \Psi}{\partial x}$, and states that the following expression must be analytic

$$W = \varphi + i\Psi \quad \text{with} \quad \frac{\mathrm{d}W}{\mathrm{d}z} = ve^{-i\theta}.$$

Stagnation point has $\boldsymbol{v} = 0$.

Uniform flow $W = (v_x - iv_y)z$

Pot flow near stagnation point $W = \frac{1}{2}kz^2$

Derivation Because at stagnation point $\boldsymbol{v} = 0$, Taylor expand $\varphi = S_{ij}\frac{x_i x_j}{2}$, then $\mathrm{div}\,\boldsymbol{v} = S_{ii} = 0$. In principal axes $\varphi = \frac{k}{2}(x^2 - y^2)$. Then $v_x = kx, v_y = -ky$ and $\Psi = kxy$. Together $W = \frac{kz^2}{2}$ (hyperbolae).

Conformal Transformations Velocities of the form $W = Az^n$, $z = re^{i\theta}$ have bounaries at $\theta = 0$ and $\theta = \frac{\pi}{n}$.

Derivation $\varphi = Ar^n \cos n\theta$, $\Psi = Ar^n \sin n\theta$. Zero flux coincides with streamlines, so $\theta = 0, \pi/n$ could be seen as boundaries.

Velocity modulus $v = |\frac{\mathrm{d}w}{\mathrm{d}z}| = n|A|r^{n-1}$ either turns to 0 ($n > 1$) or to infinity ($n < 1$).

7 Viscosity

7.1 Viscosity, Navier-Stokes Equation

Viscous stress tensor σ'_{ik} describes internal friction and is given by

$$\sigma'_{ik} = \eta\left(\frac{\partial \boldsymbol{v}_i}{\partial x_k} + \frac{\partial \boldsymbol{v}_k}{\partial x_i}\right) + \eta'\delta_{ik}\frac{\partial \boldsymbol{v}_l}{\partial x_l}.$$
$$= \eta\left(\frac{\partial \boldsymbol{v}_i}{\partial x_k} + \frac{\partial \boldsymbol{v}_k}{\partial x_i} - \frac{2}{3}\frac{\partial \boldsymbol{v}_l}{\partial x_l}\right) + \zeta\delta_{ik}\frac{\partial \boldsymbol{v}_l}{\partial x_l}$$

where in the second representation the first part is traceless.

Derivation Internal friction occurs when the fluids moves with different velocities (=gradient). We assume small gradients and hence a linear dependence. Rotational velocities should not result in internal friction, thus σ'_{ik} should depend only on symmetric combinations of spatial derivatives.

Viscosity are the coefficients η, η'. ζ is called the second viscosity.

Kinematic viscosity is the ratio $\nu = \eta/\rho$.

Navier-Stokes

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) = -\nabla p + \eta\nabla^2\boldsymbol{v} + (\eta+\eta')\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{v} \qquad \text{(Navier-Stokes)}$$

Boundary condition Navier-Stokes requires two boundary conditions. We set $\boldsymbol{v} = 0$.

Derivation Use Euler's equation in the moment flux form, add viscosity term and collect terms.

7.2 Energy Dissipation in an incompressible Fluid

Energy dissipation in an incompressible fluid

$$\frac{\mathrm{d}E_{kin}}{\mathrm{d}t} = -\frac{1}{2}\eta\int\left(\frac{\partial \boldsymbol{v}_i}{\partial x_k} + \frac{\partial \boldsymbol{v}_k}{\partial x_i}\right)^2 \mathrm{dV}$$

Derivation $\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\boldsymbol{v}^2)\right) = \rho\boldsymbol{v}\cdot\frac{\partial \boldsymbol{v}}{\partial t}$, substitute $\frac{\partial \boldsymbol{v}}{\partial t}$ from NSE, writing viscous part as $v_i\frac{\partial\sigma'_{ik}}{\partial x_k}$. Use reverse product rule trick and $\mathrm{div}\,\boldsymbol{v} = 0$ to rewrite all but one terms into $-\mathrm{div}\left(\rho\boldsymbol{v}(\boldsymbol{v}^2/2 + p/\rho) - \boldsymbol{v}\cdot\sigma'\right)$. Integrate over volume, use Gauss' theorem and use taking volume to infinity trick to kill surface integral. For $-\int\sigma'_{ik}\frac{\partial v_i}{\partial x_k}$ symmetrize velocity derivative and combine with σ'_{ik}. It follows that $\eta > 0$.

7.3 Applications

7.3.1 Viscous Flow in a Pipe

Hagen-Poiseulle

$$Q = \frac{\pi\Delta p}{8\eta l}R^4 \qquad \text{(Hagen-Poiseulle)}$$

Derivation Solve Navier-Stokes for a pipe along x-axis in cylindrical coordinates. Use $\boldsymbol{v} = (v_x(x,y),0,0)$. y,z component gives $p = p(x)$, x component gives $\frac{\mathrm{d}p}{\mathrm{d}x} = const.$, hence $\frac{\mathrm{d}p}{\mathrm{d}x} \approx \frac{\Delta p}{l}$. Integrate v in cylindrical coordinates, log part vanishes, to obtain $v = \frac{\Delta p}{4\eta l}(R^2 - r^2)$. Calculate flux $Q = \rho\int v\,\mathrm{d}^2r$ using solution.

7.3.2 Couette Flow: Flow between rotating Cylinders

Couette Flow between cylinders rotating with velocity Ω_1 (inner) and Ω_2 (outer). Velocity field $\boldsymbol{v}$ and moment of frictional forces $M_{1,2}$.

$$\boldsymbol{v} = \frac{\Omega_2R_2^2 - \Omega_1R_1^2}{R_2^2 - R_1^2}r + \frac{(\Omega_2 - \Omega_1)R_1^2R_2^2}{R_2^2 - R_1^2}\frac{1}{r}$$
$$M_1 = -M_2 = -\frac{2\pi\eta(\Omega_1 - \Omega_2)R_1^2R_2^2}{R_2^2 - R_1^2}$$

Derivation Coordinate system $v_z = v_r = 0$, $v_\varphi = v(r)$, $p = p(r)$. It holds that $\frac{\partial \boldsymbol{e}_\varphi}{\partial\varphi} = -\boldsymbol{e}_r$, $\frac{\partial^2\boldsymbol{e}_\varphi}{\partial\varphi^2} = -\boldsymbol{e}_\varphi$ and $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{\boldsymbol{v}^2}{r}\boldsymbol{e}_r$. Radial part NSE $\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\rho\boldsymbol{v}^2}{r}$, angular part $0 = \eta\nabla^2\boldsymbol{v} = \eta\left(\frac{\mathrm{d}^2\boldsymbol{v}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}r} - \frac{\boldsymbol{v}}{r^2}\right)$. Ansatz of the form r^n leads to $v = ar + \frac{b}{r}$. Use BC $v(R_{1/2}) = \Omega_{1/2}R_{1/2}$ to solve for a, b. Frictional force $f_i = -\sigma'_{ik}n_k$. Use $[\sigma'_{r\varphi}]_{r=R_1} = \eta\left[\frac{\partial v}{\partial r} - \frac{v}{r}\right]_{r=R_1} = -2\eta\frac{(\Omega_1-\Omega_2)R_2^2}{R_2^2-R_1^2}$. Total moment is found by multiplying with $2\pi R_1$.

7.3.3 River Flow

River Flow

$$p(z) = p_0 + \rho g(h-z)\cos\alpha$$
$$v(z) = \frac{\rho g\sin\alpha}{2\eta}z(2h-z)$$

Derivation Coordinate system $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = 0$, $v_x = v(z)$, $v_y = v_z = 0$, $p = p(z)$. NSE for x axis

$\frac{\mathrm{d}p}{\mathrm{d}z} + \rho g \cos\alpha = 0$, z axis $\eta \frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \rho g \sin\alpha = 0$. BC at bottom $v(0) = 0$, BC at top $\sigma_{xz}(h) = \eta \frac{\mathrm{d}v}{\mathrm{d}z} = 0$, $\sigma_{zz} = -p(h) = -p_0$

Reality check For water $\nu = \frac{\eta}{\rho} \sim 10^{-2}\,\mathrm{cm}^2\,s^{-1}$. For a rain paddle with $h = 1\,\mathrm{mm}$ we get $v \sim 5\,\mathrm{cm\,s^{-1}}$. For a slow river with $h = 10\,\mathrm{m}$, $\alpha \sim 0.1\,\frac{\mathrm{km}}{1000\,\mathrm{km}} = 10^{-4}$ we get $v(h) \sim 100\,\mathrm{km\,s^{-1}}$, which is unrealistic.

Stability check Non-linear term and hence Reynolds number vanish. How much perturbation is needed to make $\mathrm{Re} \sim 1$? $\mathrm{Re}(\beta) \sim g\frac{\alpha h^3 \beta}{\eta^2}$, where $90° - \beta$ denotes the angle between $\boldsymbol{v}$ and $\nabla \boldsymbol{v}$. For the rain paddle $\mathrm{Re}(\beta) \sim 100\beta$, for the river $\mathrm{Re}(\beta) \sim 10^{12}\beta$. The river is unstable with respect to this symmetry.

Derivation Use small angle approximation to get $\mathrm{Re}(\beta) = \frac{v(h) h \beta}{\eta}$

7.4 The Law of Similarity: Reynolds Number

Reynolds Number is the only dimensionless combination of the three parameters that determine $\boldsymbol{v}$,

$$\mathrm{Re} = \frac{uL}{\eta}. \qquad \text{(Reynolds Number)}$$

Then $\boldsymbol{v}(\boldsymbol{r}) = f(\frac{\boldsymbol{r}}{L}, \mathrm{Re})\boldsymbol{u}$.

Physical meaning is that of dominance of different terms in the Navier-Stokes equation. It holds that

$$\mathrm{Re \ large} \implies \eta \nabla^2 \boldsymbol{v} \ll \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}.$$

Similar flows are flows that can be obtained from one another by rescaling $\boldsymbol{v}$ and $\boldsymbol{r}$.

8 Laminar Flows

Laminar flows are flows where the layers of particle movements do not mix. It is characterized by a small Reynolds number.

8.1 Velocity and Pressure of laminar Flows

Velocity and pressure for flows with small Reynolds number.

$$\boldsymbol{v} = -\frac{3R}{4}\frac{\boldsymbol{u} + \boldsymbol{n}(\boldsymbol{u}\cdot\boldsymbol{n})}{r} - \frac{R^3}{4}\frac{\boldsymbol{u} - 3\boldsymbol{n}(\boldsymbol{u}\cdot\boldsymbol{n})}{r^3} + \boldsymbol{u}$$
$$p = p_0 - \frac{3}{2}\eta\frac{\boldsymbol{u}\cdot\boldsymbol{n}}{r^2}R$$

Derivation Steady NSE with low Re is $\eta\nabla^2\boldsymbol{v} = \nabla p$. Reference frame of the sphere s.t. $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{v}'$ with $\boldsymbol{v}' \to 0$ at infinity. $\operatorname{div}\boldsymbol{v} = 0 \implies \operatorname{div}\boldsymbol{v}' = 0 \implies \boldsymbol{v}' = \operatorname{rot}\boldsymbol{A}$. $\boldsymbol{A}$ must be axial and linear in $\boldsymbol{u}$, hence $\boldsymbol{A} = f'(r)\boldsymbol{n}\times\boldsymbol{u}$ with $f'(r)\boldsymbol{n} = \operatorname{grad} f(r)$. Then $\boldsymbol{v}' = \operatorname{rot}\boldsymbol{A} = \nabla\times[\nabla f(r)\times\boldsymbol{u}] = \operatorname{rot}\operatorname{rot}(f(r)\boldsymbol{u})$. Then $\operatorname{rot}\boldsymbol{v} = ... = -(\nabla^2\nabla)\times\boldsymbol{u}$. Take rot of NSE to get $0 = \nabla^2\operatorname{rot}\boldsymbol{v} = \Delta^2\nabla f\times\boldsymbol{u}$. Cannot always be parallel to $\boldsymbol{u}$ so $0 = \Delta^2 f = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\Delta f\right)$. Then $\Delta f = \frac{2a}{r} + c$ and $f = ar + \frac{b}{r}$. Take rot twice to get $\boldsymbol{v} = \boldsymbol{u} + \operatorname{rot}\operatorname{rot}(f\boldsymbol{u})$. Obtain a, b from BC $\boldsymbol{u}(r = R) = 0$. Finally $f = \frac{3Rr}{4} + \frac{R^3}{4R}$. Obtain pressure from $\operatorname{grad} p = \eta\nabla^2\boldsymbol{v} = \eta\Delta(\operatorname{grad}\operatorname{div}(f\boldsymbol{u}) - \boldsymbol{u}\Delta f) = \operatorname{grad}[\eta\Delta\operatorname{div}(f\boldsymbol{u})]$.

8.2 Stokes Formula for the Drag

Stokes Formula

$$F_x = 6\pi\eta u R \qquad \text{(Stokes Formula)}$$

Derivation *Alternative 1*: Drop non-linear term. Viscous force can then only depend on η, L, v. Use dimensional estimate to get $F \sim \nu\rho v L = \eta v L$. *Alternative 2*: On solid surface $\boldsymbol{v} = 0$ and $F_i = -\sigma_{ik}n_k = pn_i - \sigma'_{ik}n_k$. Then $F_x = \oint(-p\cos\theta + \sigma'_{rr}\cos\theta - \sigma'_{r\theta}\sin\theta)\,\mathrm{d}S$. Use $p(R) = -\frac{3\eta u}{2R}\cos\theta$, $\sigma'_{rr} = 2\eta\frac{\partial v_r}{\partial r} = 0$, $\sigma'_{r\theta} = -\frac{3\eta u}{2R}\sin\theta$ to obtain $F_x = \frac{3\eta u}{2R}\int \mathrm{d}S$.

8.3 The Layer around a moving Body

Summary of the three regions.

- *Outside the boundary layer*: Onseen equation, non-linear term cannot be neglected anymore, corrected Stokes formula, roughly potential flow
- *Within boundary layer, outside wake*: non-linear term can be neglected

- *Inside the wake*: laminar, viscosity important, vorticity, diffusion-like NSE

8.3.1 Inside the Boundary Layer

Boundary Layer is the boundary outside of which the non-linear term cannot be neglected anymore even for flows with low Reynold's number. Its boundary width is given by

$$r \ll \frac{\nu}{u}. \qquad \text{(Boundary Layer)}$$

Derivation Estimate $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \sim (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \sim \frac{u^2 R}{r^2}$ and $\nu\nabla^2\boldsymbol{v} \sim \frac{\nu u R}{r^3}$ and compare $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \ll \nu\nabla^2\boldsymbol{v}$. Alternative $\mathrm{Re} \ll 1 \iff \frac{ur}{\eta} \ll 1 \iff r \ll \frac{\eta}{u} \sim \frac{\nu}{u}$.

8.3.2 Outside the Boundary Layer

Onseen equation for flows with low Reynolds number outside of the boundary layer

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} + \nu\nabla^2\boldsymbol{v} \qquad \text{(Onseen)}$$

Derivation Approximate non-linear term.

Correction to Stokes formula for a sphere and for a cylinder moving perpendicular to its axis

$$\boldsymbol{F} = 6\pi\eta\boldsymbol{u}R\left(1 + \frac{3}{8}\mathrm{Re}\right) = 6\pi\eta\boldsymbol{u}R\left(1 + \frac{3uR\rho}{8\eta}\right)$$

$$\boldsymbol{F} = \frac{4\pi\eta\boldsymbol{u}}{\ln(3.70\nu/uR)}$$

8.3.3 Inside the laminar Wake

Wake is due to fluid particles that move along the streamlines passing close to a body. Pressure gradients force the particle around the body, but because of the internal friction it cannot fall back to its original height. The new height marks a line, the wake.

Navier-Stokes inside wake and its solution. Equation is diffusion-like

$$u\frac{\partial \boldsymbol{v}_x}{\partial x} = \nu\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}\right)\boldsymbol{v}_x$$

$$\boldsymbol{v}_x \propto \frac{1}{\nu x}\exp\left(-u\frac{z^2 + y^2}{4\nu x}\right)$$

Derivation Write NSE in x coordinate. Approximate $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \sim (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} = u_x\frac{\partial \boldsymbol{v}}{\partial x}$. Pressure doesn't change much across the wake $\implies \frac{\partial p}{\partial x} \sim 0$. Solve by switching to Fourier space. Then $u\frac{\partial v_x(k)}{\partial x} = -\eta k^2 v_x$, hence $v_x(k, x) \propto \exp(-\eta k^2 x/u)$. Upon retransforming $\boldsymbol{v}_x$ follows as above.

Transverse size is the width of the wake

$$\text{width} \sim \sqrt{\frac{\nu \cdot \text{distance from body}}{u}}$$

Derivation x is distane away from body with width y of wake. $\frac{\partial^2 v}{\partial x^2} \ll \frac{\partial^2 v}{\partial y^2}, \frac{\partial^2 v}{\partial z^2}$, hence $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \sim u\frac{\partial \boldsymbol{v}}{\partial x} \sim \frac{uv}{x}$ and $\eta\nabla^2 v \sim \eta\frac{\partial^2 v}{\partial y^2} \sim \frac{\eta v}{y^2}$. Compare to get $y \sim \sqrt{\frac{\eta x}{u}} \sim x\sqrt{\frac{\eta}{ux}} \ll x$.

Wake is laminar because $\mathrm{Re} \simeq \frac{\boldsymbol{v}_x y}{\nu} \sim x^{-1/2} \to 0$.

8.4 Drag and Lift with a Wake

Drag and lift

$$\boldsymbol{F}_x = -\rho u \iint_{Wake} \boldsymbol{v}_x \, \mathrm{d}y \, \mathrm{d}x \qquad \text{(drag)}$$

$$\boldsymbol{f}_y = \rho u \left(\int_{x_0} - \int_x\right) \boldsymbol{v}_y \, \mathrm{d}y = \rho u \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l} \qquad \text{(lift)}$$

Derivation Start by $F_i = \oint \Pi_{ik}\,\mathrm{d}S_k = \oint (p_0 + p')\delta_{ik} + \rho(u_i + v_i)(u_k + v_k)\,\mathrm{d}S_k$, where $p_0 = const.$ is pressure at infinity. Neglect constant and quadratic in v terms ($v \ll u$). Write $\left(\iint_{x_0} - \iint_x\right)\mathrm{d}y\,\mathrm{d}z \equiv \oint \mathrm{d}S_k$. Outside wake integral vanishes, because $p' \approx -\rho u v_x$ (Bernoulli), hence the integrals reduce to the wake only. For the lift use same Ansatz $\rho u\left(\iint_{x_0} - \iint_x\right) v_y\,\mathrm{d}y\,\mathrm{d}z$. Add constant vanishing integrals at $y = \pm const$ to make it a line integral. $\boldsymbol{F}_y = \int \boldsymbol{f}_y\,\mathrm{d}z$.

Lift of wing explained with $v_2 > v_1 \implies p_2 < p_1$. However fluid particles do not meet again at the end of the wing.

9 Turbulent Flows

9.1 Symmetry Breaking

Symmetries are broken with increasing Reynolds number in the following order:

1. Left-right symmetry is broken,
2. time invariance discretizes, i.e. solution become periodic,
3. up-down symmetry is spontaneously broken (von Karman vortex street),
4. z-axis translation symmetry is broken,
5. flows become chaotic,
6. symmetries are restored in a statistical sense.

9.2 Instabilities

Instabilities occur when small perturbations amplify. For solutions $\boldsymbol{v}_1 = A(t)\boldsymbol{v}_1(r)$, we can make a Landau expansion

$$\frac{\mathrm{d}|A|^2}{\mathrm{d}t} = 2\gamma_1|A|^2 - \alpha|A|^4 - \dots$$

and get for $\alpha > 0$

$$|A|_{max} \propto (\mathrm{Re} - \mathrm{Re}_{critical})^{1/2}.$$

For $\alpha < 0$, we add a term of sixth order $-\beta|A|^6$ and get

$$|A|_{max} \propto \frac{|\alpha|}{2\beta} \pm \sqrt{\frac{\alpha^2}{4\beta^2} + 2\frac{\gamma_1}{\beta}}.$$

Derivation Steady solution $\boldsymbol{v}_0(\boldsymbol{r})$, small perturbation $\boldsymbol{v}_1(\boldsymbol{r}, t)$. Pressure $p = p_0 + p_1(\boldsymbol{r}, t)$. Substitute into NSE, linearize (drop $(\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_1$) to get the Eigen value problem $\frac{\partial \boldsymbol{v}_1}{\partial t} + (\boldsymbol{v}_0 \cdot \nabla)\boldsymbol{v}_1 + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}_0 = -\frac{\nabla p_1}{\rho} + \nu\Delta\boldsymbol{v}_1$ and $\operatorname{div} \boldsymbol{v}_1 = 0$ with BC $\boldsymbol{v}_1 = 0$. Fourier series $\boldsymbol{v}_1 = \sum \boldsymbol{v}_\omega(r)e^{-i\omega t} = \boldsymbol{A}(t)v_1(\boldsymbol{r})$. Unstable if for one $\Im\omega > 0$. Write $A(t) \propto e^{\gamma_1 t - \imath\omega_1 t}$ for short times (after that: saturation). Expand $\frac{\mathrm{d}|\boldsymbol{A}|^2}{\mathrm{d}t} = 2\gamma_1|\boldsymbol{A}|^2 + -\alpha|\boldsymbol{A}|^4 + O(6)$ (odd terms vanish from averaging). Solve for $\alpha > 0$ and maximize. Expand γ_1 near Re_c s.t. $\gamma_1 \propto \mathrm{Re} - \mathrm{Re}_c$. For $\alpha < 0$ add $-\beta|\boldsymbol{A}|^6$ term.

Kelvin-Helmholtz instability occur between two tangential layers traveling with different velocities. Small asymmetries lead to larger/smaller velocities. This leads to to smaller/larger pressure. This makes the velocities even larger/smaller and so on.

9.3 Developed Turbulence

Behavior at large Re **is important** because for example already a small river has $\mathrm{Re} \simeq 10^6$.

Mean dissipation energy Energy dissipation remains constant in the limit $\mathrm{Re} \to \infty$ although $\nu \to 0$. It is given by

$$\epsilon = \langle \nu(\nabla_\alpha \boldsymbol{v}_\beta)^2 \rangle = \langle \boldsymbol{v} \cdot \boldsymbol{f} \rangle \sim \frac{u^3}{R}$$

where R is the radius of the body.

Derivation *Estimate*: Fluid with large Re, body with radius R. During time $\tau \sim \frac{R}{u}$ body gets momentum $p \sim \rho R^3 u$ from fluid. Drag force $F \sim \frac{p}{\tau} \sim \rho R^2 u^2$. Then $\epsilon = \frac{Fu}{\rho R^3}$.
Quantitative: Add random force $\frac{\boldsymbol{f}(\boldsymbol{r},t)}{\rho}$ to NSE with $\langle f_\alpha(t, \boldsymbol{r}) f_\beta(t', \boldsymbol{r}') \rangle = \delta(t - t')\chi_{\alpha\beta}(\boldsymbol{r} - \boldsymbol{r}')$. Multiply NSE by $\boldsymbol{v}$ and integrate to obtain $\frac{\partial}{\partial t} \int \frac{\boldsymbol{v}^2}{2} \mathrm{d}^d r = -\nu \int (\nabla_\alpha \boldsymbol{v}_\beta)^2 \mathrm{d}^d r + \int \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}^d r$ = - dissipation + energy injection. For stationary state mean energy is constant.

Energy cascade picture Energy is injected into large scale motion $\sim L$ (energy containing scale). Large eddies break into smaller and even smaller eddies without loss of energy. These tiny eddies at viscous scale $\sim \lambda$ dissipate energy (dissipative scale). The ratio L/λ grows as Re increases.

9.4 Kolmogorov Theory of developed Turbulence

Scale dependent Reynolds number $\mathrm{Re}_l = \frac{v_l l}{\nu}$. Viscosity becomes important for $\mathrm{Re}_\lambda \sim 1$.

Initial range is the region $\lambda \ll r \ll L$. Assumption is that all properties are independent on viscosity.

Kolmogorov Obukov law relates velocity variations over distances

$$\Delta v(l) \sim (\epsilon l)^{1/3}. \qquad \text{(Kolmogorov Obukov)}$$

Separation of two point of fluid grows with time as

$$\delta l^2(t) \propto t^3. \qquad \text{(Richardson)}$$

Dissipative scale λ is given by $\lambda \sim \frac{L}{\mathrm{Re}^{4/3}} \ll L$.

Derivation KO law: From dimensional estimate $\epsilon \sim \frac{(\delta u)^3}{l}$. Richardson: Use $l^3 \sim v^3 t^3 \sim \epsilon l t^3$. Scale: Write $\mathrm{Re}_l \sim \frac{\Delta v(l) l}{\nu} \sim \dots \sim \frac{(\epsilon L)^{1/3}}{\nu} L \left(\frac{l}{L}\right)^{4/3} \sim \mathrm{Re}\left(\frac{l}{L}\right)^{4/3}$. Use $\mathrm{Re}_\lambda \sim 1$ to determine λ.

Kármán–Howarth equation $\frac{\partial}{\partial t}\langle v(x)v(y)\rangle = \frac{1}{2}\nabla_x \langle (\boldsymbol{v}(x) - \boldsymbol{v}(y))(\boldsymbol{v}(x) - \boldsymbol{v}(y))^2 \rangle - 2\nu \langle \nabla_\alpha \boldsymbol{v}_\beta(x) \cdot \nabla_\alpha \boldsymbol{v}_\beta(y) \rangle + \chi_{\alpha\alpha}(\frac{x-y}{L})$

Derivation TODO

Kolmogorov's 4/5 law is given by $S_3^{||}(r) = -\frac{12}{d(d+2)}\epsilon r$. In 3D it becomes

$$S_3^{||}(r) = -\frac{4}{5}\epsilon r$$

i still dont know what $S_3^{||}(r)$ is

Derivation TODO

9.5 Intermittency

TODO

9.6 The Energy Spectrum

TODO

10 Waves

10.1 Gravity Waves

We are interested in the potential φ such that $\boldsymbol{v} = \nabla\varphi$ and the dispersion relations.

Equation to solve is

$$\nabla^2\varphi = 0, \quad \left(\frac{\partial\varphi}{\partial z} + \frac{1}{g}\frac{\partial^2\varphi}{\partial t^2}\right)_{z=0} = 0$$

with boundary condition $\frac{\partial\varphi}{\partial z}|_{z=-h} = 0$.

Derivation Neglect non-linear term: if the amplitude $a \ll \lambda$ (estimate as $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \sim \frac{v^2}{\lambda} \sim \frac{va}{\lambda\tau}$ and $\frac{\partial v}{\partial t} \sim \frac{v}{\tau}$). Assume incompressible pot flow, i.e. $\operatorname{rot}\boldsymbol{v} = \operatorname{div}\boldsymbol{v} = 0$, hence $\nabla^2\varphi = 0$. Rewrite Euler's without non-linear and φ, s.t. there is a ∇ before all terms. Kill ∇ to get expression for the pressure. Then at surface $p_0 = -\rho\left(g\xi + \frac{\partial\varphi}{\partial t}\right)$. Redefine $\varphi \to \varphi + p_0 t/\rho$. Take time derivative, use $v_z = \frac{\partial\xi}{\partial t}$ and $v_z = \frac{\partial\varphi}{\partial z}$.

Deep water Trajectories are circles, dispersion is non-linear.

$$\varphi = Ae^{kz}\cos(kx - \omega t), \quad \omega = \sqrt{gk}$$

Shallow water Trajectories are ellipses, dispersion is linear $\omega^2 = kg\tanh(kh) \approx ghk^2$ for $kh \ll 1$.

$$\varphi = A\cosh(k(z+h))\cos(kx - \omega t), \quad \omega = \sqrt{ghk}$$

Derivation Use Ansatz $\varphi = f(z)\cos(kx - \omega t)$. Then $\nabla^2\varphi = 0$ results in $f(z) = e^{kz}$ for deep water (BC $f \to 0$ for $z \to -\infty$) and $f(z) = \cosh(k(z+h))$ for shallow water (BC $f'(z=-h) = 0$). Second eq results in dispersion rel $\omega^2 = gk$.

Damping of gravity waves Amplitude of wave decreases as $\exp(-\gamma t)$, where $\gamma = \frac{2\nu\omega^4}{g^2}$ is the damping coefficient.

Derivation Change in energy $\frac{\mathrm{d}E}{\mathrm{d}t} = -2\eta\int\left(\frac{\partial^2\varphi}{\partial x_i\partial x_k}\right)^2\mathrm{d}V = -2\eta\int(\varphi_{xx}^2 + \varphi_{zz}^2 + 2\varphi_{xz}^2)\,\mathrm{d}V$. Use averaging $\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\frac{\mathrm{d}E}{\mathrm{d}t}\,\mathrm{d}t = -8\eta k^4\int\overline{\varphi^2}\,\mathrm{d}V$. Average energy $\overline{E} = \int\rho\overline{v^2}\,\mathrm{d}V = 2\rho k^2\int\overline{\varphi^2}\,\mathrm{d}V$. Damping coefficient $\gamma = \frac{\dot{\overline{E}}}{2\overline{E}} = 2\nu k^2 = \frac{2\nu\omega^4}{g^2}$.

Alternative: Dimensional guess for dispersion relation. For deep water, the parameters are ω, k, g. For shallow water, h additionally.

Circle expansion $\omega^2 = gk \iff r = gt^2$: the circles expand with the acceleration of the free fall.

10.2 Dispersive Waves

Circular Waves on the Deep Water the deformation and the radius of the n-th ripple are

$$\xi \propto \frac{gt^2}{r^3}\cos\left(\frac{gt^2}{4r}\right), \quad r_n = \frac{gt^2}{8\pi n}$$

Derivation Use Ansatz $\varphi \propto e^{kz}e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t}$. Use $\nabla^2\varphi = 0$, $g\frac{\partial\varphi}{\partial z} + \frac{\partial^2\varphi}{\partial t^2} = 0$ to derive dispersion $\omega^2 = gk$. Integrate all $\boldsymbol{k}$ to get $\varphi(z=0) \approx \zeta = \int e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\sqrt{gk}t}\,\mathrm{d}^2k = \int e^{ikr\cos\theta - i\sqrt{gk}t}k\,\mathrm{d}k\,\mathrm{d}\varphi$. Use method of stationary phase $f(t) = \int e^{ith(x)}\,\mathrm{d}x \approx \sqrt{\frac{2\pi}{t|h''(x_0)|}}e^{ith(x_0)}e^{i\frac{\pi}{4}\operatorname{sgn}(h''(x_0))}$. Determine $h(k)$, calculate derivatives and pluck into solution first for k integral, then for φ integral to obtain sol above.

Group velocity is given by $\frac{\partial\omega}{\partial\boldsymbol{k}}$ sucht that

$$\varphi(\boldsymbol{r}, t) = e^{i\boldsymbol{k}\cdot\boldsymbol{r} - \omega(\boldsymbol{k})t}f\left(\boldsymbol{r} - \frac{\partial\omega}{\partial\boldsymbol{k}}t\right)$$

Derivation Use Ansatz $\varphi = e^{i\boldsymbol{k}\cdot\boldsymbol{r}}f(\boldsymbol{r})$ with $f(\boldsymbol{r}) = \sum_{q\ll k}f_q e^{i\boldsymbol{q}\cdot\boldsymbol{r}}$ slowly varying. Taylor $\omega(\boldsymbol{k}+\boldsymbol{q}) = \omega(k) - \frac{\partial\omega}{\partial\boldsymbol{k}}\boldsymbol{q}$ to obtain $\varphi(\boldsymbol{r}, t) = e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega(\boldsymbol{k})t}f\left(\boldsymbol{r} - \frac{\partial\omega}{\partial\boldsymbol{k}}t\right)$.

Kevin angle: Ship Waves have a group velocity of $v_{gr} = \frac{1}{2}\sqrt{\frac{g}{k}}$. Then the maximal angle θ_0 of the ship wave cone is $\theta_0 \approx 19.5°$ (Kelvin angle).

Derivation Sum over all rings $h \propto \int_{-\infty}^{0} h_t \, \mathrm{d}t$, $h_t \propto \exp(iu(t))$, $u = gt^2/4r(t)$. Determine r from triangle including source of wave, position of ship now and position of interest, then $r(t) = \sqrt{R^2 + v^2t^2 + 2Rvt\cos\theta}$, where θ is the angle and R the distance between the ship now and the point of interest. Use method of stationary phase (largest contribution comes from region near extremum) $0 = \dot{u} = ... \propto v^2t_3^2 Rvt\cos\theta + 2R^2$. Roots negative for $\sin\theta < \sin\theta_0 = 1/3$.

Capillary Waves are surface waves that take the change of surface energy into account. The generalized wave equation and its dispersion relation are

$$0 = \left[\rho g \frac{\partial\varphi}{\partial z} + \rho\frac{\partial^2\varphi}{\partial t^2} - \alpha\frac{\partial}{\partial z}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right)\right]_{z=0}$$

$$\omega^2 = gk + \frac{\alpha}{\rho}k^3.$$

For $k \ll k_* = \sqrt{\rho g/\alpha}$ we get capillary waves with $\omega^2 = \alpha k^3/\rho$.

Derivation Molecules at surface have higher energy. Change of surface $S = \int \sqrt{1 + (\nabla^2\zeta)^2}\,\mathrm{d}x\,\mathrm{d}y \approx \int 1 + \frac{1}{2}(\nabla\zeta)^2\,\mathrm{d}^2r$, then $\delta S = \int \nabla\zeta\nabla\delta\zeta\,\mathrm{d}^2r = -\int \nabla^2(\zeta)\delta\zeta\,\mathrm{d}^2r$. Balance change of surface energy by pressure $\alpha\delta S - \int p\delta\zeta\,\mathrm{d}S = 0$ to get $p = -\alpha\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2}\right)$. Add to equation from gravity waves $p = -\rho(g\zeta + \frac{\partial\varphi}{\partial t})$. Take time derivative from obtained expression and use $\frac{\partial\zeta}{\partial t} = v_z = \frac{\partial\varphi}{\partial z}$ to get the above generalized wave equation. Use Ansatz $\varphi = Ak^{kz}\cos(kx - \omega t)$ to obtain dispersion.

Rayleigh-Taylor or: why does water pour out of overturned glass? $w(k) = \sqrt{-gk + \frac{\alpha}{\rho}k^3}$ is imaginary for small $k \sim \frac{1}{r}$, where r is the radius of the glass, and unstable with respect to ripple formation.

10.3 Sound

Equation of sound waves for adiabatic motions with small rel changes $p = p_0 + p'$, $\rho = \rho_0 + \rho'$ with constant eq p_0, ρ_0 and $p' \ll p_0$, $\rho' \ll \rho_0$.

$$\frac{\partial^2 X}{\partial t^2} - c^2\nabla^2 X = 0 \quad \text{with} \quad c = \sqrt{\left(\frac{\partial p}{\partial\rho}\right)_S}$$

where $X = \rho', \varphi, \boldsymbol{v}, p$.

Dispersion for sound waves obeying the above equation is linear

$$\omega = ck.$$

Derivation Assume small relative changes, then formulae for p, ρ follow. Continuity eq $\frac{\partial\rho'}{\partial t} + \rho_0 \operatorname{div}\boldsymbol{v} = 0$, Euler's eq $\frac{\partial\boldsymbol{v}}{\partial t} + \frac{1}{\rho_0}\nabla p' = 0$ (oscillations are small, non-linear term drops out). For adiabatic motion $p' = \left(\frac{\partial p}{\partial\rho}\right)_S \rho'$. Take $\frac{\partial}{\partial t}$ of continuity eq, div of Euler's eq and combine to obtain wave equation for ρ'.

Solution for 1D sound wave is $\varphi = f_1(x - ct) + f_2(x + ct)$

Sound waves are longitudinal $\boldsymbol{v} = \operatorname{grad}\varphi$, only v_x non-zero $\implies \boldsymbol{v} \parallel \boldsymbol{k}$.

Sound wave pressure variation $p' = \rho_0 vc \iff \frac{\rho'}{\rho_0} = \frac{v}{c}$ is larger than in an incompressible flow where from Bernoulli $p' \sim \rho_0 v^2 \iff \frac{\rho'}{\rho} = \left(\frac{v}{c}\right)^2$.

Derivation For $\varphi = f(x - ct)$, $v = \frac{\partial\varphi}{\partial x} = f'(x - ct)$, $p' = -\rho_0\frac{\partial\varphi}{\partial t} = \rho_0 cf'(x - ct)$. Equate f' and use $p' = c^2\rho'$. Note that we assume $\rho' \ll \rho$.

Isothermal speed of waves $\left(\frac{\partial p}{\partial\rho}\right)_S = \gamma\left(\frac{\partial p}{\partial\rho}\right)_T$ with $\gamma = \frac{c_p}{c_v}$. For $p = nk_BT = \frac{\rho k_B T}{m}$ we get $c = \sqrt{\gamma\frac{k_BT}{m}}$.

Motion is adiabatic if the displacement during one period of oscillation is much less than the wavelength of the oscillation, i.e. $l \ll \lambda$.

Derivation If molecules move diffusion like with velocity v_{th}, then $\langle R^2\rangle \simeq v_{th}lt$, where l is the mean free pass. Thermal equilibrium is slow if $v_{th}lT = \langle R^2\rangle \ll \lambda^2$, then $v_{th}l \ll c\lambda$ and use $c \simeq v_{th}$.

Spherical Wave obey the equation of motion $\frac{\partial\varphi}{\partial t^2} = \frac{c^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)$ with general solution $\varphi = \frac{f_1(r-ct)}{r} + \frac{f_2(r+ct)}{r}$. Amplitude decreases as $1/r$, intensity as $1/r^2$.

Sound in a moving Medium has the dispersion relation

$$\omega = c|\boldsymbol{k}| + \boldsymbol{u}\cdot\boldsymbol{k}$$

and velocity of propagation $\frac{\partial\omega}{\partial\boldsymbol{k}} = \frac{c\boldsymbol{k}}{k} + \boldsymbol{u}$.

Derivation Consider moving reference frame K and system moving with the fluid K'. Then $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{u}t$. Insert into Ansatz $\varphi \propto \exp(i\boldsymbol{k}\cdot\boldsymbol{r}' - ikct)$.

Doppler Effect, moving observer $\omega = ck - \boldsymbol{u}\cdot\boldsymbol{k} = \omega_0(1 - \frac{u}{c}\cos\theta)$.

Doppler Effect, moving source $\omega = \frac{\omega_0}{1-\frac{u}{c}\cos\theta}$.

Derivation *Moving observer*: K' of source (system at rest) with frequency $\omega_0 = kc$. K system moving with observer fluid has velocity $-\boldsymbol{u}$. Thus $\omega = ck - \boldsymbol{u}\cdot\boldsymbol{k}$.
Moving source: K' of source (system moving) with frequency $\omega_0 = ck(1 - \frac{u}{c}\cos\theta)$, fluid moves with velocity $-\boldsymbol{u}$. K system of observer (at rest) has $\omega = ck$. Thus $\omega = \frac{\omega}{1-\frac{u}{c}\cos\theta}$.

A Vector Identities & Indices

Vector Identitites

$$\begin{aligned} \text{rot grad}\,\boldsymbol{v} &= 0 \\ \text{div rot}\,\boldsymbol{v} &= 0 \\ \text{rot rot}\,\boldsymbol{v} &= \text{grad div}\,\boldsymbol{v} - \nabla^2\boldsymbol{v} \\ (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} &= \nabla\left(\frac{\boldsymbol{v}^2}{2}\right) - \boldsymbol{v}\times\text{rot}\,\boldsymbol{v} \end{aligned}$$

Index notations

$$\begin{aligned} \text{div}\,\boldsymbol{v} &= \frac{\partial \boldsymbol{v}_k}{\partial x_k} \\ [\text{grad}\,f]_i &= \frac{\partial f}{\partial x_i} \\ [(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}]_i &= \boldsymbol{v}_k\frac{\partial \boldsymbol{v}_i}{\partial x_k} \\ [\text{div grad}\,\boldsymbol{v}]_i &= \frac{\partial}{\partial x_i}\left(\frac{\partial \boldsymbol{v}_l}{\partial x_l}\right) \\ [\nabla^2\boldsymbol{v}]_i = [(\nabla\cdot\nabla)\boldsymbol{v}]_i &= \frac{\partial^2 \boldsymbol{v}_i}{\partial x_k \partial x_k} \end{aligned}$$

$\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{A}(\nabla\cdot\boldsymbol{B}) - \boldsymbol{B}(\nabla\cdot\boldsymbol{A}) + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B}$

B Tricks

Some reoccurring tricks used in derivations.

B.1 Partial Integration

B.2 Taking surfaces to Infinity

At infinity there are usually no deformations $\boldsymbol{u}_i$, hence integrals like $\oint \sigma_{ik}u_i\,\mathrm{d}\boldsymbol{S}_k$ vanish.

B.3 Reverse Product Rule

$g\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}fg}{\mathrm{d}x} - \frac{\mathrm{d}g}{\mathrm{d}x}f$

B.4 Geometric Identities

$\boldsymbol{n}\parallel\boldsymbol{z} \implies \sigma_{iz} = 0$
$\boldsymbol{u} = \boldsymbol{u}(x,y), \boldsymbol{u}\parallel\boldsymbol{z} \implies \text{div}\,\boldsymbol{u} = 0 \implies \Delta u_z = 0$

B.5 Integration of cylindical Equation

Expressions like $\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) = A = const.$ are integrates as $f = \frac{1}{4}Ar^2 + B\log r + C$, where B, C are integration constants.

B.6 Both terms need to vanish independently

If expressions like $\delta a(...) + \delta a\cdot b(...) = 0$ need to hold for all δa.

C Math Shit I should know, but don't

C.1 Spherical Coordinates

$$\begin{aligned} \nabla f =& \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\boldsymbol{e}_\varphi \\ \Delta f =& \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) \\ &+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} \\ \nabla\cdot\boldsymbol{v} =& \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(v_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} \\ \nabla\times\boldsymbol{v} =& \frac{1}{r^2}\left(\frac{\partial(v_\varphi\sin\theta)}{\partial\theta} - \frac{\partial v_\theta}{\partial\varphi}\right)\boldsymbol{e}_r \\ &+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\varphi} - \frac{\partial r v_\varphi}{\partial r}\right)\boldsymbol{e}_\theta \\ &+ \frac{1}{r}\left(\frac{\partial r v_\theta}{\partial r} - \frac{\partial v_r}{\partial\theta}\right)\boldsymbol{e}_\varphi \\ \mathrm{d}\boldsymbol{l} =& \mathrm{d}r\boldsymbol{e}_r + r\,\mathrm{d}\theta\boldsymbol{e}_\theta + r\sin\theta\,\mathrm{d}\varphi\boldsymbol{e}_\varphi \\ \mathrm{d}\boldsymbol{S} =& r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\boldsymbol{e}_r + r\sin\theta\,\mathrm{d}r\,\mathrm{d}\varphi\boldsymbol{e}_\theta + r\,\mathrm{d}r\,\mathrm{d}\theta\boldsymbol{e}_\varphi \\ \mathrm{d}V =& r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi \end{aligned}$$

C.2 Cylindrical Coordinates

$$\begin{aligned}
\nabla f =& \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\boldsymbol{e}_\varphi + \frac{\partial f}{\partial z}\boldsymbol{e}_z \\
\Delta f =& \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} \\
\nabla \cdot \boldsymbol{v} =& \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} \\
\nabla \times \boldsymbol{v} =& \left(\frac{1}{r}\frac{\partial v_z}{\partial \varphi} - \frac{\partial v_\varphi}{\partial z}\right)\boldsymbol{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\boldsymbol{e}_\varphi \\
& + \frac{1}{r}\left(\frac{\partial r v_\varphi}{\partial r} - \frac{\partial v_r}{\partial \varphi}\right)\boldsymbol{e}_z \\
\mathrm{d}\boldsymbol{l} =& \mathrm{d}r\boldsymbol{e}_r + r\,\mathrm{d}\varphi\boldsymbol{e}_\varphi + \mathrm{d}z\boldsymbol{e}_z \\
\mathrm{d}\boldsymbol{S} =& r\,\mathrm{d}\varphi\,\mathrm{d}z\boldsymbol{e}_r + \mathrm{d}r\,\mathrm{d}z\boldsymbol{e}_\varphi + r\,\mathrm{d}r\,\mathrm{d}\varphi\boldsymbol{e}_z \\
\mathrm{d}V =& r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z
\end{aligned}$$