Game Theory

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1 Cooperative Games

1.1 Definition

Cooperative Game

- **players**: $N = \{1, 2, \ldots, n\}$ (finite)
- **coalitions**: $C \subseteq N$ form in the population and become players resulting in coalition structure $\rho = \{C_1, C_2, \ldots, C_k\}$.
- **payoffs**: $\phi = \{\phi_1, \ldots, \phi_n\}$ something like $\phi_i = \phi(\rho,$ "sharing rule").

Characteristic function form (CFG) game

- **game**: A CFG defined by tupel $G(v, N)$
- **players**: $N = \{1, 2, \ldots, n\}$ (finite, fixed population)
- **coalitions**: disjoint $C \subseteq N$ form resulting in coalition structure/ partition $\rho = \{C_1, C_2, \ldots, C_k\}$.
- **characteristic function**: $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, is the characteristic function form that assigns a worth $v(C)$ to each coalition
- **outcome**: partition $\rho = \{C_1, C_2, \ldots, C_k\}$ directly implies a payoff allocation/ imputation $\phi_i = f_i(C_i)$. There are no side-payments and the worth of a coalition cannot be (re-)distributed.

1.2 The Core

Assumption (Superadditivity) *If two coalitions C, S are disjoint, then*

$$v(C) + v(S) \leq v(C \cup S),$$

i.e. mergers of coalitions weakly improve their worths.

Corollary 1.1 *The grand coalition is efficient, i.e. $\sum_{C \in \rho} v(C) \leq v(C)$.*

Definition 1.2 (The Core) *of a superadditive $G(v, N)$ consists of all outcomes where the grand coalition forms and payoff allocations ϕ^* are*

- *pareto-efficient, i.e. $\sum_{i \in N} \phi_i^* = v(N)$ ("nothing should be wasted"),*
- *unblockable, i.e. for all $C \subset N$, $\sum_{i \in C} \phi_i^* \geq v(C)$.*

Properties of the core A system of weak linear inequalities defines the core, it is therefore closed and convex. The core can be empty, non-empty, large. The core is somewhat both, descriptive and normative.

Definition 1.3 (Balancedness) *Let $\alpha : 2^N \to [0, 1]$ assign a balancing weight to any $C \in 2^n$. A set of balancing weights is a balanced family if, for every i, $\sum_{C \in 2^n : i \in C} \alpha(C) = 1$. Then, a superadditive game is called **balanced** if for all balanced families $\sum_{C \in 2^N} \alpha(C) v(C) \leq v(N)$.*

Theorem 1.4 (Bondareva-Shapley) *The core of a cooperative game is nonempty if and only if the game is balanced.*

Limitations of the core The core may be empty, non-empty, but inequitable (landowner gets everything), or large (any split of 1).

1.3 Shapley Value

Axiom (Shapley value) *Given some $G(v, N)$, an acceptable allocation/ value $x^*(v)$ should satisfy*

- ***Efficiency*** *$\sum_{i \in N} x_i^* = v(N)$*
- ***Symmetry*** *If, for any two players i, j, $v(S \cup i) = v(S \cup j)$ for all S not including i, j, then $x_i^* = x_j^*$.*
- ***Dummy player*** *If, for any i, $v(S \cup i) = v(S)$ for all S not including i, then $x_i^* = 0$.*
- ***Additivity*** *If u, v are two characteristic functions, then $x^*(u + v) = x^*(u) + x^*(v)$.*

Theorem 1.5 (Shapley Value) *The unique function satisfying all four axioms for the set of all games is*

$$\phi_i(v) = \underbrace{\sum_{S \subset N : i \in S} \frac{(|S| - 1)!(n - |S|)!}{n!}}_{\text{average operator (number of orders)}} \underbrace{\left(v(S) - v(S \backslash \{i\})\right)}_{\text{marginal contribution}}.$$

Is it $S \subset N$ or really $S \in N$, which I think wouldn't make sense? *The shapley value is always unique, feasible, payable, but may not be in the core.*

Axiom (Young) *A set of equivalent axioms is*

- ***Efficiency*** *$\sum_{i \in N} x_i^* = v(N)$*
- ***Symmetry*** *If, for any two players i, j, $v(S \cup i) = v(S \cup j)$ for all S not including i, j, then $x_i^* = x_j^*$.*
- ***Monotonicity*** *If u, v are two characteristic functions and, for all S not including i, $u(S) \geq v(S)$, then $x_i^*(u) \geq x^*(v)$.*

Relationship Core and Shapley value None. Shapley value is normative, Core is hybrid. When Core is nonempty, the Shapley value may lie inside or not. When the Core is empty, the Shapley value is still uniquely determined.

Non-transferable-utility cooperative game

- **game**: A CFG defined by tupel $G(v, N)$
- **players**: $N = \{1, 2, \ldots, n\}$ (finite, fixed population)
- **coalitions**: disjoint $C \subseteq N$ form resulting in coalition structure/ partition $\rho = \{C_1, C_2, \ldots, C_k\}$.
- **characteristic function**: $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, is the characteristic function form that assigns a worth $v(C)$ to each coalition
- **outcome**: partition $\rho = \{C_1, C_2, \ldots, C_k\}$ and payoff allocation $\phi = \{\phi_1, \ldots, \phi_n\}$. Each coalition's payoff $v(C_j)$ may be shared among all players $i \in C_j$ (**transfer of utils**) such that for all $j = 1, \ldots, k$ the share is **feasible**, i.e. $\sum_{i \in C_k} \phi_i \leq v(C)$.

Agents have preferences over coalitions.

1.4 Matching Problem

Diese Section ist noch nicht fertig, weil wir sie nicht verstanden haben.

matching problem

from NTU tu TU matching was bedeutet diese folie? bzw was bedeutet NTU und TU allgemein? inwiefern passt das matching problem zu NTU? inwiefern haben wir vorher nur TUs behandelt?

Theorem 1.6 (Gale-Shapley) *For any marriage problem, one can make all matchings stable using the deferred acceptance algorithm.*

2 Non-cooperative Games

2.1 Definition

Non-cooperative Game

- **players**: $N = \{1, 2, \ldots, n\}$ (finite)
- **actions/ strategies**: each player chooses s_i from his own finite strategy set, S_i for each $i \in N$. Resulting strategy combination: $s = (s_1, \ldots, s_n) \in (S_i)_{i \in N}$.
- **payoffs**: $u_i = u_i(s)$ resulting from the outcome of the game determined by s.

2-player games

- **Prisoner's dilemma**: social dilemma, tragedy of the commons, free-riding. Conflict between individual and collective incentives.
- **Harmony**: aligned incentives. No conflict between individual and collective incentives.
- **Battle of the sexes**: coordination. Conflict and alignment of individual and collective incentives.
- **hawk dove/snowdrift**: anti-coordination. Conflict and alignment of individual and collective incentives.
- **Matching pennies**: zero-sum, rock-paper-scissor. Conflict of individual incentives.

2.2 Overview of Solution Concepts

Solution concept is a formal rule for predicting how a game will be played. These predictions are called solutions and describe, which strategies will be adopted by players and therefore the results of the game.

Definition 2.1 (Equilibrium/ solution) *is a rule that maps the structure of a game into an equilibrium set of strategies s^*.*

Definition 2.2 (Best response) *Player i's best response (or, reply) to the strategies s_{-i} played by all others is the strategy $s_i^* \in S_i$ such that*

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \forall s_i' \text{ and } s_i' \neq s_i^*.$$

Definition 2.3 ((Pure strategy) Nash equilibrium) *All strategies must be best responses*

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \forall s_i' \text{ and } s_i' \neq s_i^*.$$

Braess' Paradox 60 people travel from S to D over either A or B. The nash equilibrium without a middle road is that 30 people travel over A and 30 people travel over B. With the middle road, the nash equilibrium is such that all people travel from S to A, then the middle route to B and from there to D. The total travel time worsens to 120 min.

3 Preferences and Utility

3.1 Preferences

Definition 3.1 (Binary relation) *$\succeq$ on a set X of decision alternatives for a player is a non-empty subset $P \subset X \times X$. We might write $x \succeq y$ if and only if $(x, y) \in P$ with interpretation that the player weakly prefers x over y. Let similarly $x \succ y$ denote player's strict preference of x over y and $x \sim y$ an indifference between x and y.*

3.2 Modern Assumptions on Preferences

Axiom (Completeness) *$\forall x, y \in X : x \succeq y$ or $y \succeq x$ or both.*

Discussion Completeness For Consumers/ agents/ humans it is hard to rank options, because decision making takes time and effort and the agents might be uninformed, uncertain, unable to evaluate what a product is and does, subject to biases/ inattention. Examples were the Chinese vegetables.

Axiom (Transitivity) *$\forall x, y, z \in X$: if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.*

Discussion Transitivity Consumers/ agents/ humans often find it difficult to rank choices coherently. Ranking depends on many dimensions (speed, space, etc) and different needs. Example were the three cars ranked.

Axiom (Continuity) *Let $\succeq$ be a rational preference ordering on X. For $x \in X$ define the set of alternatives that are (weakly) worse/ better tan x*

$$W(x) = \{y \in X : x \succeq y\}, \quad B(x) = \{y \in X : y \succeq x\}.$$

Then $B(x), W(x)$ are closed sets for all $x \in X$.

3.3 Utility

Definition 3.2 (Utility function) *for a binary relation $\succeq$ on a set X is a function $u : X \to \mathbb{R}$ such that*

$$u(x) \geq u(y) \iff x \succeq y.$$

Proposition 3.3 (Existence 1) *There exists a utility function for each complete, transitive, positively measurable, and continuous preference ordering on any closed set.*

Proposition 3.4 (Existence 2) *There exists a utility function for every transitive and complete preference ordering on any countable set.*

3.4 Expected-utility theory

Utility $\neq$ Payoff Game: if head turns up at n-th toss you win n CHF. Then the expected payoff $\mathbb{E}[\text{lottery}] = \infty$. From Bernoulli's suggestion of **diminishing marginal utility** of wealth and the need for utility characterization under uncertainty, this lay the foundation for expected utility theory.

wieso folgt aus diesem game über theory of diminishing marginal utility, dass utility $\neq$ payoff und damit expected utility theory + was hat expected utility theory mit independence of irrelevant alternatives zu tun?

Setting Let $T = \{\tau_1, \ldots, \tau_m\}$, $\tau_i \in \mathbb{R}^m$ such that $(\tau_i)_j = \delta_{ij}$, $i = 1, \ldots, m$, be a finite set and let X consist of all probability distributions on T:

$$X = \triangle(T) = \{x = (x_1, \ldots, x_m) \in \mathbb{R}_+^m : \sum_{k=1}^m x_k = 1\}$$

Axiom (Independence of irrelevant alternatives)

$$\forall x, y, z \in X, \ \forall \lambda \in (0,1) : x \succ y \implies (1-\lambda)x + \lambda z \succ (1-\lambda)y + \lambda z$$

Example 3.5 *Agent prefers Asian food over Italian, but occasionally likes Italian. Then sushi $\succ$ pizza $\not\Longrightarrow$ $(1-\lambda)$ sushi + λ wontons $\succ$ (1λ) pizza + λ wontons.*

Discussion Independence of irrelevant alternatives It must be possible for any decision to be broken down into its smallest parts, i.e. it is reductionist, because the axiom always compares only two decisions.

Bernoulli function/ von Neumann-Morgenstern utility function If $\succeq$ is a binary relation on X representing the agent's preference over lotteries over T. If there is a function $\nu : T \to \mathbb{R}$ such that

$$x \succeq y \iff \sum_{k=1}^m x_k \nu(\tau_k) \geq \sum_{k=1}^m y_k \nu(\tau_k)$$

then

$$u(x) = \sum_{k=1}^m x_k \nu(\tau_k)$$

defines a utility function for $\succeq$ on X.

Theorem 3.6 (von Neumann-Morgenstern) *Let $\succeq$ be a complete, transitive and continuous preference relation on $X = \triangle(T)$, for any finite set T. Then $\succeq$ admits a utility function u of the expected-utility form if and only if $\succeq$ meets the axiom of independence of irrelevant alternatives.*

Allais paradox Sets such that people tend to choose $x_1 \succeq x_2$ and $x_3 \succeq x_4$, but it can be shown that $x_1 \succeq x_2$ implies $x_4 \succeq x_3$, which is a contradiction.

Sure thing principle A decision maker who would take a certain action if he knew that event B happens and also if he knew that not B happens, should also take the same action if he knew nothing about B. In easy: would you take an action independent of knowing B, do it.

Lemma 3.7 (Characterization of sure thing principle) *Assume that everything the decision maker knows is true then sure thing principle is equivalent to independence of irrelevant alternatives.*

3.5 Comparability

Theorem 3.8 (Translation invariance) *Given a Bernoulli function ν for given preferences $\succeq$, let $\mu' = \alpha + \beta\nu$, where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^+$. Then ν' is also a Bernoulli function for another utility function*

$$u' = \alpha + \beta u.$$

That is, expected utility functions are unique up to a positive affine transformation.

Ordinality and cardinality

- **Ordinal utility function**: only comparisons $u(x) \geq u(y)$ are meaningful, e.g. she likes x less than y.
- **Cardinal utility function**: also differences $u(x) - u(y)$ are meaningful, e.g. she likes x over y twice as much as y over z.

- **Util**: fundamental measure of utility (util) that is not invariant to any transformations, e.g. she likes x five times more than y.

Interpersonal comparability (IC) is a utility function, where utility differences between players make "sense". Ordinal and cardinal utility function are possible IC, utils are IC. For example, a cardinal utility function that is IC under some non-affine increasing transformation is still IC, but not cardinal anymore. However, comparing utilities between agents is almost always impossible and implies some welfare statement/ judgment. can we compare utils or not?

was ist nicht IC? beispiel und gegenbeispiel für IC ordinal und cardinal functions?

3.6 Risk

Definition 3.9 (Lottery) *TODO*

Definition 3.10 (Risk-neutrality) *An agent is risk-neutral if and only if he is indifferent between accepting and rejecting all fair gambles, that is for all α, τ_1, τ_2*

$$\mathbb{E}[u(lottery)] = \alpha v(\tau_1) + (1-\alpha)v(\tau_2) = u(\alpha\tau_1 + (1-\alpha)\tau_2).$$

An agent is risk-neutral if and only if he has a linear von Neumann-Morgenstern utility function.

wieso folgt die zweite gleichheit? was ist $u(\alpha\tau_1 + (1-\alpha)\tau_2)$?

Definition 3.11 (Risk-aversion) *An agent is risk averse if and only if he rejects all fair gambles, that is for all α, τ_1, τ_2*

$$\mathbb{E}[u(lottery)] = \alpha v(\tau_1) + (1-\alpha)v(\tau_2) < u(\alpha\tau_1 + (1-\alpha)\tau_2),$$

which is a strictly concave utility function.

Definition 3.12 (Risk seeking) *An agent is risk seeking if and only if he strictly prefers all fair gambles, that is for all α, τ_1, τ_2*

$$\mathbb{E}[u(lottery)] = \alpha v(\tau_1) + (1-\alpha)v(\tau_2) > u(\alpha\tau_1 + (1-\alpha)\tau_2),$$

which is a strictly convex utility function.

why is that convex? aren't u and v interchanged?

Remarks If you believe that people have preferences, under "reasonable" axioms we can translate them into utility functions. Nevertheless, we should always be aware that our analysis is based on several assumptions/ axioms. Money is not equal to utility (recall diminishing marginal utility). Preferences do not have to be self regarding ("homo economicus").

4 Normal Form Games

4.1 Definition

Definition 4.1 (Normal form game) *A normal form (or strategic form) game consists of three objects.*

- *Players: $N = \{1, \dots, n\}$ with typical player $i \in N$*
- *Strategies: For every player i, a finite set of strategies, S_i, with typical strategy $s_i \in S_i$.*
- *Payoffs: a function $u_i : (s_1, \dots, s_n) \to \mathbb{R}$ mapping strategy profiles to a payoff for each player i, and an overall mapping $u : S \to \mathbb{R}^n$.*

Thus, a normal form game is represented by the triplet $G = \langle N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$.

Definition 4.2 (Strategy profile) *A set $s = (s_1, \dots, s_n)$ is called a strategy profile. Is is a collection of strategies, one for each player. If s is played, player i receives $u_i(s)$.*

Definition 4.3 (Opponents strategies) *Write s_{-i} for all strategies expect for the one of player i. So a strategy profile may be written as $s = (s_i, s_{-i})$.*

4.2 Rationality and Dominance

Definition 4.4 (Dominance) *of strategies over other strategies.*

- ***Strict dominance**: a strategy s_i strictly dominantes s_i' if $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for all s_{-i}.*
- ***Weak dominance**: a strategy s_i weakly dominantes s_i' if $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all s_{-i}.*
- ***Dominated strategy**: a strategy s_i' is strictly dominated if the is an s_i that strictly dominates it.*
- ***Dominant strategy**: A strategy s_i is strictly dominant if it strictly dominates all $s_i' \neq s_i$.*

If players are rational they should never play a strictly dominated strategy, no matter what others are doing. They may play weakly dominated strategies.

Definition 4.5 (Dominant-Strategy Equilibrium) *The strategy profile s^* is a dominant-strategy equilibrium if, for every player i, $u_i(s^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for all strategy profiles $s = (s_i, s_{-i})$.*

Assumption of Rationality assumes that players are rational decision makers and that mutual rationality is common knowledge, that is: I know that she knows that I will play rational, she knows that "I know that she knows that I will play rational", and so on. Further suppose that all players know the game and that again is common knowledge.

Iterative deletion of strictly dominated strategies If the game and rationality of players are common knowledge, iterative deletion of strictly dominated strategies yields the set of "rational" outcomes.

Battel of sexes as example with no dominated strategy

5 Equilibria

Definition 5.1 (Nash Equilibrium) *is a strategy profile s^* such that for every player i*

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \quad \forall s_i.$$

At s^*, no i regrets playing s_i^*. Given all the other players' actions, i could have not done better. No player can improve utility by changing strategy.

Definition 5.2 (Best-reply Function) *for player i is a function B_i such that*

$$B_i(s_{-i}) = \{s_i | u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \ \forall s_i'\}.$$

Definition 5.3 (Nash Equilibrium) *A strategy s^* is a Nash equilibrium if and only if $s_i^* \in B_i(s_{-i}^*)$ for all i.*

A Nash Equilibrium is a strategy profile of mutual best responses each player picks a best response to the combination of strategies the other players pick.

Game	#	Nash-equilibrium	Comment
Prisoner's dilemma	1	both confess	Worst outcome
Harmony	1	(cooperate, cooperate)	Best outcome
Battle of the sexes	2	(Boxing, Boxing), (Shopping, Shopping)	Good outcome, both pareto optimal
hawk dove/snowdrift	2	(Hawk, Dove), (Dove, Hawk)	Both pareto optimal, but perhaps (Dove, Dove) "better"
Matching pennies	0		

Pareto optimality is a measure of efficiency or optimality. An outcome of a game is Pareto optimal if there is no other outcome that makes every player at least as well off and at least one player strictly better off. Pareto optimality is not a solution concept.

5.1 Mixed Strategies

Definition 5.4 *A mixed strategy σ_i for a player i is any probability distribution over his or her set S_i of pure strategies. The set of mixed strategies is*

$$\triangle(S_i) = \left\{ x_i \in \mathbb{R}_+^{|S_i|} : \sum_{h \in S_i} x_{ih} = 1 \right\}.$$

Definition 5.5 (Mixed extension) *of a game G has players, strategies and payoffs $\Gamma = \langle N, \{S_i\}_{i\in N}, \{U_i\}_{i \in N}\rangle$, where strategies are probability distribution in the set $\triangle(S_i)$ and U_i is player i's expected utility function assigning a real number to every strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$.*

Definition 5.6 (Expected utility function)

$$U_i(\sigma) = \sum_s u_i(s) \prod_{j \in N} \sigma_j(s_j)$$

Definition 5.7 (Opponent's strategies) *σ_{-i} is a vector of mixed strategies, one for each player, except i. So $\sigma = (\sigma_i, \sigma_{-i})$.*

Definition 5.8 (Best-reply function) *for player i is a function β_i such that*

$$\beta_t(\sigma_{-i}) = \{\sigma_i | U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i})\ \forall \sigma_i'\}.$$

Best-reply graph TODO

Definition 5.9 (Mixed-strategy Nash Equilibrium) *is a profile σ^* such that*

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i, \sigma_{-i}^*) \quad \forall \sigma_i, i.$$

Proposition 5.10 *$x \in \Delta(S)$ is a Nash equilibrium if $x \in \beta(x)$.*

p31/48 was heisst das $s_i \in supp(x_i) \implies s_i \in \beta(x)$?

sind expected utility und U_i gleich?

Theorem 5.11 (Nash's existence) *Every finite game has at least one [Nash] equilibrium in mixed strategies.*

A strictly dominated pure strategy cannot play a part in a Nash equilibrium

Theorem 5.12 (Wilson) *Generically, any finite normal form game has an odd number of Nash equilibria.*

"Generically" means that if you slightly change payoffs the set of Nash equilibria does not change. For example consider continua of nash equilibria, perturbing payoffs slightly (usually?) results in an odd number of nash equilibria only, which makes the theorem still hold.

Proposition 5.13 (Nash equilibria are perserved by transformation) *Any two games G, G', which differ only by a positive affine transformation of each player's payoff function have the same set of Nash equilibria.*

Adding a constant c to all payoffs of some player i which are associated with any fixed pure combination s_i for the other players sustains the set of Nash equilibria.

Remarks Nash equilibrium is a powerful concept since it exists (in finite settings). But there are often a multitude of equilibria. Therefore game theorists ask which equilibria are more or less likely to be observed. We will focus next on a static refinements, strict and perfect equilibrium. Later we will talk about dynamic refinements.

5.2 Equilibria Refinements

Definition 5.14 (Strict Nash Equilibrium) *is a profile σ^* such that $U_i(\sigma_i^*, \sigma_{-i}^*) > U_i(\sigma_i, \sigma_{-i}^*)$ for all σ_i, i.*

Definition 5.15 (ε-perfection) *Given any $\varepsilon \in (0,1)$, a strategy profile σ is ε-perfect if it is interior ($x_{ih} > 0$ for all $i \in N$ and $h \in S_i$) and such that*

$$h \notin \beta_i(x) \implies x_{ih} \leq \varepsilon$$

Definition 5.16 (Perfect equilibrium) *A stratefy profile σ is perfect if it is the limit of some sequence of ε_t-perfect strategy profiles x^t with $\varepsilon_t \to 0$.*

trembling hand perfection, wie kann man sich das intuitiv vorstellen und wie kann man es effizient überprüfen?

Proposition 5.17 (Selten) *For every finite game there exists at least one perfect equilibrium. The set of perfect equilibria is a subset of the set of Nash equilibria.*

Proposition 5.18 *Every strict equilibrium is perfect.*

6 Dynamic Games

Definition 6.1 (Extensive-form Game) *is defined with the following.*

- *Players, $N = \{1, \ldots, n\}$, with typical player $i \in N$ (Nature can be one of these players).*
- *Basic structure is a tree, the **game tree** with nodes $a \in A$. Let a_0 be the root of the tree.*
- *Nodes are game states which are either **decision nodes**, where some player chooses an action, **chance nodes**, where nature plays according to some probability distribution.*

Extensive form is a directed graph with single initial node. Edges represent moves, probabilities on edges represent Natures moves. Nodes that the player in question cannot distinguish (**information sets**) are circled together (or connected by a dashed line).

Extensive and normal forms A strategy is a player's complete plan of action, listing move at every information set of the player. Different extensive form games may have same normal form (loss of information on timing and information).

Number of player's strategies is the number of actions available at each of his information set.

Subgames Given a node a in the game tree consider the subtree rooted at a. The node a is the root of a subgame if

- a is the only node in its information set
- if a node is contained in the subgame then all of its successors are also contained in the subgame.
- every information set in the game either consists entirely of successor nodes to a or contains no successor node to a.

what is an information set?

können so nicht keine knoten roots von subgames sein, vor denen ein information set gelagert ist?

Definition 6.2 (Strategies in extensive games)

- ***Pure strategy** s_i: one move for each information set of the player*
- ***Mixed strategy** σ_i: any probability distribution x_i over the set of pure strategies S_i.*
- ***Behavior strategy** y_i: select randomly at each information set the move to be made. Moves are made with independent probabilities at information sets.*

what are behavior strategies?

37/47: "The indicated outcome, with probabilities in brackets, results from the mixed strategy", was bedeutet das? wieso folgt daraus, dass es keine behavior strategy gibt?

Definition 6.3 (Perfect recall) *Player i in an extensive form game has perfect recall if for every information set h of player i, all nodes in h are preceded by the same sequence of moves of player i.*

Definition 6.4 (Realization equivalent) *A mixed strategy σ_i is a realization equivalent with a behavior strategy y_i if the realization probabilities under the profile σ_i, σ_{-i} are the same as those under y_i, σ_{-i} for all profiles σ.*

Theorem 6.5 (Kuhn) *Consider a player i in an extensive form with perfect recall. For every mixed strategy σ_i there exists a realization-equivalent behavior strategy y_i.*

Definition 6.6 (Subgame Perfect Equilibrium) *A behavior strategy profile in an extensive form game is a subgame perfect equilibrium if for each subgame the restricted strategy is a Nash equilibrium of the subgame.*

what does this restriction look like in the case of the Outside-option game when the information set in the subgame is split? Before we have strategies aa, ab, ba, bb, do we have a, b afterwards or still aa, ab, ba, bb?

Theorem 6.7 *Every finite game with perfect recall has at least one subgame perfect equilibrium. Generic such games have a unique subgame perfect equilibrium, where generic means that with probability 1 when payoffs are drawn from continuous independent distributions.*

Backward-induction First consider last decision and find Nash equilibria, then use this information to consider what to do at the second-to-last time and find the Nash equilibria there for each previous Nash. Continue until the root of the game is reached.

2 more games

- **Outside-option game** which is battle of the sexes, but player 1 can decide if she joins the game before. There exist three subgame perfect equilibria, one for each equilibrium of the BS game: $\{EA, A\}, \{TB, B\}, \{T(3/4A + 1/4B), 1/4A + 3/4B\}$.
- **Centiped game** with unique subgame perfect equilibrium stop at all nodes.

7 Evolutionary Game Theory

Approaches in economics.

- The rationalistic paradigm in economics. A person's behavior is based on maximizing some goal function (utility) under given constraints and information.
- The "as if" approach. Do not theorize about the intentions of agents' actions, but consider only the outcome (observables). Similar to the natural sciences where a model is seen as an approximation of reality rather than a causal explanation.

Mass-action interpretation A large population of identical individuals represents each player role in a game. The game is played recurrently. In each period one individual from each player population is drawn randomly to play the game. Individuals observe samples of earlier behaviors in their own population and avoid suboptimal play (successful strategies are copied more frequently). Nash's claim is that if all individuals avoid suboptimal pure strategies and the population distribution is stationary then it constitutes a [Nash] equilibrium.

Theorem 7.1 (Folk) *If the population process converges from an interior initial state, then for large t the distribution is a Nash equilibrium. If a stationary population distribution is stable, then it coincides with a Nash equilibrium.*

what is a distribution in this context? bzw. was ist ein interior initial state?

Definition 7.2 (Symmetric two player normal form game) $G = \langle N, \{S\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ *consists of three objects*

- *Players:* $N = \{1, 2\}$ *with typical player* i.
- *Strategies:* $S_1 = S_2 = S$ *with typical strategy* $s \in S$.
- *Payoffs: A function* $u_i : (h,k) \to \mathbb{R}$ *mapping strategy profiles to a payoff for each player* i *such that* $u_2(h,k) = u_1(k,h)$ *for all* $h,k \in S$.

Definition 7.3 (Symmetric Nash equilibrium) *is a strategy profile* σ^* *such that for every player* i

$$u_i(\sigma^*, \sigma^*) > u_i(\sigma, \sigma^*) \quad \forall\sigma.$$

Proposition 7.4 *In a symmetric normal form game there always exists a symmetric Nash equilibrium.*

Not all Nash equilibria of a symmetric game need to be symmetric.

Definition 7.5 (Evolutionary stable Strategy) *A mixed strategy* $\sigma \in \Delta(S)$ *is an evolutionary stable strategy (ESS) if for every strategy* $\tau \neq \sigma$ *there exists* $\varepsilon(\tau) \in (0,1)$ *such that for all* $\varepsilon \in (0, \varepsilon(\tau))$

$$U(\sigma, \varepsilon\tau + (1-\varepsilon)\sigma) > U(\tau, \varepsilon\tau + (1-\varepsilon)\sigma).$$

Let Δ^{ESS} *be the set of evolutionary stable strategies.*

would probably be better to add an player index to the utility function U_i, although the game is symmetric (makes life easier, for example what is $\geq$ for a vector?)

Proposition 7.6 (Characterization of ESS) *A mixed strategy* $\sigma \in \Delta(S)$ *is an evolutionary stable strategy (ESS) if*

$$U(\tau,\sigma) \leq U(\sigma,\sigma) \quad \forall\tau$$
$$U(\tau,\sigma) = U(\sigma,\sigma) \Rightarrow U(\tau,\tau) < U(\sigma,\tau) \quad \forall\tau \neq \sigma.$$

Theorem 7.7 (Relations) *If* $\sigma \in \Delta(S)$ *is weakly dominated, then it is not evolutionary stable. If* $\sigma \in \Delta^{ESS}$*, then* (σ,σ) *is perfect equilibrium. If* (σ,σ) *is a strict Nash equilibrium, then* σ *is evolutionarily stable.*

Relations of Equilibria Strict Nash equilibrium $\subseteq$ Evolutionary stability $\subseteq$ Perfect equilibrium $\subseteq$ Nash equilibrium

Notes Evolutionary game theory studies mutation processes (ESS). The stable states often coincide with solution concepts from the "rational" framework. Evolutionary game theory does not explain how a population arrives at such a strategy. This is studied in behavioral game theory.

8 Interactive Environments and distributed Control

Characteristics of distributed control applications Multiple decision making agents, interdependency, no central authority, distributed information, collective performance. This constitutes a game.

Centralized vs. distributed control Distributed information may be costly in communication, it's not just "multi component" or "graph structure", and may result in efficiency loss due to tragedy of commons or the price of Anarchy.

Distributed efficiency loss results from the difference in objective, i.e. local objectives differ from collective objectives. The price of Anarchy quantifies the system's ratio of centralized and decentralized performance.

Game theory and distributed control have a lot in common. One may even call distributed control the study of Game theory without the machinery of Game theory.

8.1 Descriptive Agenda

what is descriptive and prescriptive agenda?

Descriptive agenda tries to describe and model: stylized models of societal situations, emphasis on new insights, not necessarily design tool, design only once behavior understood

?? The foundational assumptions are rationality, perception and evolution. For example take the beauty game. Rationality results in an Nash equilibrium of 0, however, player's will play 1/2 of what the expect others to play, i.e. perception. This may change over time, which is evolution.

Learning/ evolutionary Games shift the focus away from solution concepts, i.e. Nash equilibrium, towards how players might arrive at solutions, i.e. dynamics. With distributed control, we now take the opposite approach. Instead of describing a solution, we now identify a target and dynamics that lead to this target.

8.2 Prescriptive agenda

Prescriptive agenda ??? was ist hier der entscheidende punkt? 22/47

8.3 Learning Rule

Near far search algorithm for bees Bees fly to different patches of flowers foraging for nectar. If nectar per flower is abundant (high payoffs), bees continue in the current patch with high probability. If a series of flowers yields low payoff, bees fly far away to a new patch. This is a successful strategy at the population level, implementing a total payoff maximizing Nash equilibrium.

Single turbine modeled as game:

- Players $i = 1, 2, \ldots, n$ (windmills/ turbines)
- Finite strategy set $A_i = \{a_i, b_i, \ldots, k_i\}$ (orientations)

- Joint strategy space $A = \prod_i A_i$ (wind park configuration)
- Payoffs $u_i : A \to \mathbb{R}$ (own energy production)

How do we get the windmills to play this game - giving them private utility functions - so as to maximize total energy production?

Algorithm 1 Learning Rule

Initialize twice: each turbine i selects a
random (benchmark) orientation $\overline{a}_i^{0,1}$
resulting in power $u_i^{0,1}$
for $t > 1$ **do**
  if $\overline{a}_i^t \neq \overline{a}_i^{t-1}$ or $u_i^t \geq u_j^{t-1}$ **then**
    windmill 'content'
  if $\overline{a}_i^t = \overline{a}_i^{t-1}$ and $u_i^t < u_i^{t-1}$ **then**
    windmill 'discontent'
  if 'content' and $u_i^t \geq u_i^{t-1}$ **then**
    keep benchmark, i.e. $\overline{a}_i^t = \overline{a}_i^t$
  if 'content' and $u_i^t < u_i^{t-1}$ **then**
    switch benchmark , i.e. $\overline{a}_i^t = \overline{a}_i^{t-1}$
  if 'discontent' **then**
    keep old benchmark, i.e. ???
  if 'content' **then**
    windmills play $\overline{a}_i^t$ with probability $1 - \varepsilon$
    RAND with ε
  else
    windmills play RAND with probability 1

Theorem 8.1 *For any desired probability $p < 1$, there exists $\varepsilon > 0$ such that, for sufficiently large iterations, total power generated is maximal with at least probability p.*

Intuition A series of experiments leads to state with even higher welfare until someone's payoff goes down. That individual becomes discontent and his searching may cause other agents to become discontent. Eventually, the discontent agents settle into a new all-content state, where the settling probability increases with overall welfare of the state.

8.4 Cooperative Control

Single turbine modeled as game:

- Players $i = 1, 2, \ldots, n$ (windmills/ turbines)
- Finite strategy set $A_i = \{a_i, b_i, \ldots, k_i\}$ (orientations)
- Joint strategy space $A = \prod_i A_i$ (wind park configuration)
- Payoffs $u_i : A \to \mathbb{R}$ (total energy production)

Making windmills play this game – giving them altruistic utility functions – will also maximize total energy production.

Transforming games by adding altruism First add constant to make all payoffs positive. Then, for each cell, sum payoffs of both players $u_i'(s_i) = \sum_j u_j(s_j)$. For example, $u_1(A, A) = 3$, $u_2(A, A) = 1$, then $u_1'(A, A) = 3 + 1 = 4$ and $u_2'(A, A) = 1 + 3 = 4$. This transforms the prisoners dilemma into the harmony game with a unique Nash equilibrium. However, note that this does not always work.

Difference in information

- **Own energy production**: $u_i(\phi_s) = \phi_s$. No information necessary about structure of the game, program dynamic offline, specific dynamics will work, dynamic requires no feedback.
- **Total energy**: $u_i(\phi_s) = \phi_s + \phi_0$. Need to understand structure of the game in order to identify which specification will generate desired equilibria, more general class of dynamics will work, program dynamics offline, dynamic requires feedback about energy total as game continues.

Which approach is better depends on the situation.

9 Experimental Game Theory

Behavioral economics Experiments by Allais/ Ellsberg/ Kahneman-Tversky challenge axioms of standard decision theory and with it the notion of man as a "perfectly rational" expected utility maximizer. The clean "theory of expected utility" contradicted by these simple experiments lead to behavioral economics.

Experiments on animal behavior carried out by by Thorndike 1898, Morgan 1903, Pavlov 1927, Thorpe 1956 show "follow the path of success/ avoid the path of failure", later formalized as "radical behaviorism"/ "reinforcement learning".

Homo economicus as the perfect rationality, pure self-interest straw man.

- **Perfect rationality**: common knowledge, common beliefs, optimization
- **Pure self-interest**: narrow self interest, no concern for others' payoffs, no consideration of one's actions, decisions are not subject to social influence.

More realistic environment

- **Knowledge and information**: unknown game structure, etc
- **Behavior and motivations**: behavioral heuristics

9.1 Experiments

9.1.1 Ultimatum game

Description One side proposer moves first: makes a proposal as how to split a cake. The other side recipient responds and either accepts the offer so that it will be realized, or destroys the cake (both get zero).

Nash equilibria are any proposals made, where the responder accepts.

Subgame perfection Proposer takes all, accept nevertheless.

Studied with the Ultimatum game Nash equilibrium (responder should always accept), subgame perfection (proposer gives nothing), reputation models in case of repetition, social preferences such as fairness, pro-sociality, spitefulness.

Information settings

- high information: players know the structure of the game, know their own position in the game, know the payoff structure, the game is anonymous.
- low information: players do not know the payoff structure of the game, do not observe others' actions, learn only about payoffs as they realize.

Discussion The unique subgame-perfect Nash equilibrium is an extreme allocation. Any rejection by the responder kills own and other's payoff, while any positive proposal, presuming (rational) acceptance, seems like a gift. However, presuming (off the equilibrium-path) rejection of low offers, a substantial proposal may be strategically rational. Hence, it may be rational to have a rejection reputation.

Experiments Meta-analysis suggests: proposals of roughly 40 %, high rejection rates for proposals under 20%, intermediate rejection rates for proposals of 20%-40%, and almost zero rejection rates for proposals $> 40\%$. Over time, decline or no decline of proposals depending on experimental/matching protocol.

9.1.2 Public Goods Game

Description Contributions are socially valuable (increase total payoffs as $R > 1$), but each individual has an incentive the withhold his own contribution (free-ride as $R/n < 1$). The payoff functions is given by $\phi_i(c) = (B - c_i) + \sum_{j \in N} mcpr \cdot c_j$.

Nash equilibrium universal non-contribution.

Studied with the Public Goods game Nash equilibrium, social preferences such as fairness, pro-sociality, conditional cooperation, reciprocity, mechanisms such as punishment, rewards etc.

Information settings

- high information: players know the structure of the game, know their own position in the game, know the payoff structure, the game is anonymous.

- low information: players do not know the payoff structure of the game, do not observe others' actions, learn only about payoffs as they realize.

Discussion Again, the Nash equilibrium is an extreme allocation Lowest social welfare Pareto-dominated by social optimum. Any positive contribution decreases own payoff but increases those of others and increases total welfare.

Experiments Meta-analysis suggests: average contributions of roughly 40%-50% when game is played once or in the first round when repeated; when repeated (with random re-matching without any mechanism): over time, contributions roughly halve every 10-20 periods depending on matching protocol.

9.2 Interpretations

9.2.1 Subjective Utility Correction Project

The failure to play according to Nash equilibrium as predicted by pure self-interest is explained using alternative payoff functions that include social preferences and concerns for other players' payoffs such as

- Fairness considerations (Fehr-Schmidt)
- Inequality/inequity aversion (Bolton-Ockenfels)
- Altruism (Fehr-Gachter, Gintis-Bowles-Boyd-Fehr, Fehr-Fischbacher)
- Reciprocity (Fischbacher-Gachter-Fehr)

Note: This approach (by the Zurich school) mirrors the various "corrections" to utility functions motivated by ambiguity aversion, etc.

Social personas In the one-shot game and in the final period of a repeated game, he would contribute zero. However, if his utility contains a concern for the other player, and is, for example, Cobb-Douglas of the form

$$u_i(c) = \phi_i^{1-\alpha_i} \cdot \phi_{-i}^{\alpha_i}$$

where ϕ_{-i}^{α} is the average payoff to players $j \neq i$, then we have a range of personas depending on α

- 0 rational
- $(0, 0.5)$ moderate altruist
- 0.5 impartial altruist
- $(0.5, 1)$ strong altruist
- 1 pure altruist

9.2.2 Mistakes equilibrium

The failure to play according to Nash equilibrium as predicted by pure self-interest is explained by relaxing the rationality assumption. Examples of such models include "Noise"/ QRE (Palfrey-Prisbey) and intuitive versus contemplative players (Rubinstein).

Summary According to such a model, positive contributions are evidence of "less" or bounded rationality.

9.2.3 Learning

The failure to play according to Nash equilibrium as predicted by pure self-interest is explained by adaptive processes of learning to play the game. Examples of such models include Reinforcement learning (Roth-Erev), Directional learning (Selten), Perturbed best reply (Young), Belief-based learning (Fudenberg-Levine), and EWA (Camerer-Ho).

9.3 Testing Interpretations

9.3.1 Overall Analysis

Experiment setup Experiments involving 236 subjects in 16 sessions. In each session, 16 players played four of our games. The mpcr was 0.4 or 1.6 The budget was 40 coins each round. Each game was repeated for 20 rounds. Players received instructions containing different amounts of information about the game and sometimes (anonymous) feedback about previous-period play. Play was incentivized with real money (e.g. one coin=0.01 CHF).

By design of the experiment, games differed with respect to whether contributing zero was a strictly dominant strategy. In half of the games, contributing everything was a strictly dominant HOE strategy (e.g. by setting the mpcr = 1.6 ¿ 1). In the other half of the games, contributing nothing was a strictly dominant HOE strategy (e.g. by setting the mpcr = 0.4 ¡ 1).

Summary of experimental results In total, there therefore are

- 46.7% players consistent with homo oeconomicus,
- 15.4% are consistent and anti-social,
- 21.4% are consistent and pro-social,
- 16.5% are inconsistent, meaning pro-social in one and anti-social in the other — mistakes.

The median is neutral, the mean close to neutral. Note that inconsistent players in terms of social preferences may by consistent in terms of 'erroneous play'

9.3.2 Analysis by Amount of Information

Types of information

- Black box: Players do not know the structure of the game, learn nothing about other players' actions or payoffs, and know their own history of actions and payoffs only.
- Standard (enhanced): Players know the structure of the game, and learn what others did in the past as the game is repeated. (Players are explicitly told what payoffs others got).

9.3.3 Learning

A simple model of learning Suppose players initially make random contributions. Thereafter, they follow the direction of payoff increases and they avoid the direction of payoff decreases. Notice such a learning rule is completely uncoupled (Foster and Young 2006) from information about others' actions and payoffs, relying only on own realized payoffs.

Conditional cooperation Suppose players contribute/free-ride if others do too (Fischbacher et al, EL 2001). They increase their contributions if others increase their contributions, and they decrease their contributions if others decrease their contributions. Notice such a learning rule is uncoupled (Hart and Mas-Colell 2003) from information about others' payoffs, relying only on own realized payoffs and others' actions.

A richer black box learning model Suppose players initially make random contributions. Thereafter, adjustments follow four regularities:

- Asymmetric inertia: stay with your current strategy more often after success than after failure
- Search volatility: search for new strategies more randomly after failure than after success
- Search breadth: search for new strategies further away after failure than after success
- Directional bias: follow the direction of payoff increases, and avoid the direction of payoff decreases

9.4 Summary

Theory vs. reality Mainstream game theory relies on rather extreme assumptions such as complete information, common knowledge, unbounded rationality, and optimizing behavior. In many real-world situations, these assumptions are untenable because the game may be too complex, behavior of others may be unobservable, players may not know others' utility functions, and the structure of the game may be unknown. In addition, real-world humans care about others, and follow certain rules/norms.

Experiments Play often does not coincide with the Nash equilibrium predictions. There are robust deviations from predictions, and many experiments have made similar observations. To explain these deviations, we must abandon the assumption of narrow self-interest in favor of social preferences and/or abandon the assumption of strictly optimizing behavior in favor of behavior that allows for heuristics/learning.

Learning Over time, play approaches equilibrium in most settings, including those where very limited information is available. There is a rich theoretical literature on these convergence properties, but relatively little of it has been tested in the laboratory. And there is a lack of acknowledgement in experimental research of the fact that simple heuristics may explain behavior not only in low-information but also in richer information environments. There is plenty of room for innovative experimental-theoretical work in this area.