Fokker-Planck equation

Janik Schuettler, Adrian Rutschmann

Motivation: Brownian particle

A Brownian particle is subject to random forces, for example fat globules in milk or pigments in watercolor.

- What does random mean?
- How do we describe the non-deterministic evolution in time of the particle?
- What are the connections to quantum mechanics?

Table of contents

Table of Contents

Probability basics: Random variables

Random variable:

A random variable X is a function $\Omega \rightarrow \mathbb{R}^n$ from the sample space Ω into the reals. For a Brownian particle we have for example

$$X: \text{ set of all trajectories} \rightarrow \mathbb{R} \quad \{x(t)\} \mapsto x(t_1)$$

Probability density:

The probability density function $\mathcal{P}(x_1, ..., x_n)$ captures the probability that the random variable has a value within the volume element $dx_1...dx_n$ at $(x_1, ..., x_n)$

Probability basics: Moments and correlations

Moments:

For a random variable X, we may compute statistical moments

$$\langle X^n \rangle := \int dx\, x^n \mathcal{P}(x).$$

Correlations:

The correlation of two random variables X and Y is

$$\langle XY \rangle = \int dxdy\, xy \mathcal{P}(x, y),$$

where $\mathcal{P}(x, y)$ is the probability of finding X at x *and* Y at y

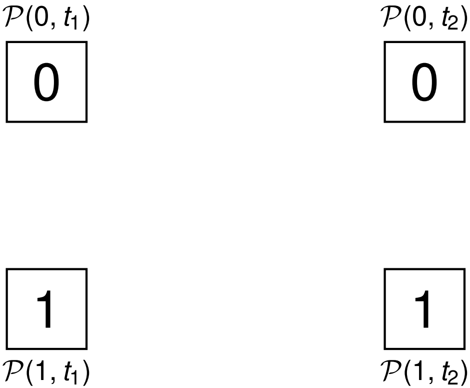
Stochastic processes

A stochastic process is a set of random variables $y(t)$ where t denotes the time.

A stochastic process is characterized by

- The probability density $\mathcal{P}(y_1, t_1; y_2, t_2; \ldots; y_n, t_n)$.
- The conditional probability $\mathcal{P}(y_n, t_n | y_1, t_1; y_2, t_2; \ldots; y_{n-1}, t_{n-1})$.

Conditional probability



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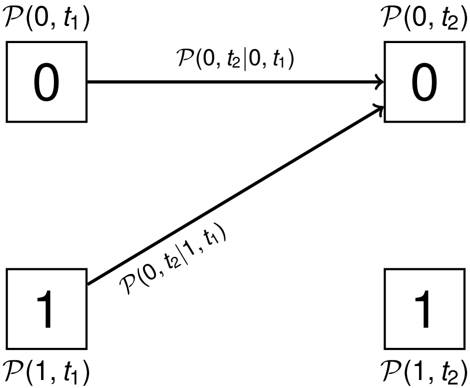
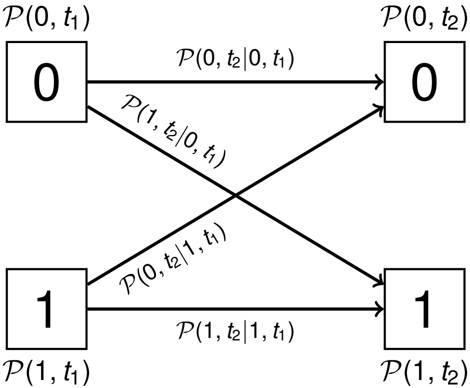


Evolution of the system:

$$\mathcal{P}(0, t_2) = \mathcal{P}(0, t_2|0, t_1)\mathcal{P}(0, t_1) + \mathcal{P}(0, t_2|1, t_1)\mathcal{P}(1, t_1)$$

Conditional probability



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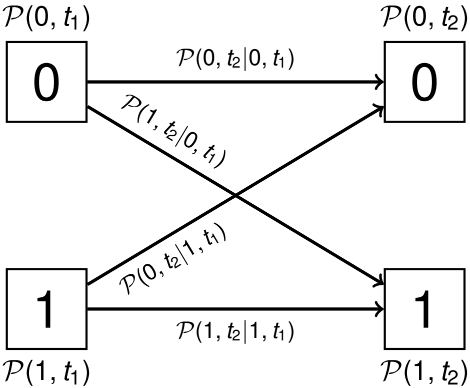

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We have to go somewhere:

$$\mathcal{P}(0, t_2|0, t_1) + \mathcal{P}(1, t_2|0, t_1) = 1$$

$$\mathcal{P}(0, t_2|1, t_1) + \mathcal{P}(1, t_2|1, t_1) = 1$$

Conditional probability

In a continuous version we get the normalization condition

$$1 = \mathcal{P}(1, t_2|1, t_1) + \mathcal{P}(2, t_2|1, t_1) \quad \rightarrow \quad 1 = \int \mathrm{d}y \, \mathcal{P}(y, t_2|x, t_1)$$

and the evolution equation

$$\mathcal{P}(0, t_2) = \mathcal{P}(0, t_2|0, t_1)\mathcal{P}(0, t_1) + \mathcal{P}(0, t_2|1, t_1)\mathcal{P}(1, t_1)$$
$$\rightarrow \mathcal{P}(y_2, t_2) = \int \mathrm{d}y_1 \, \mathcal{P}(y_2, t_2|y_1, t)\mathcal{P}(y_1, t)$$

Markov processes

A Markov process has *no memory*:

"Probability to change depends only on where I am now"

$$\mathcal{P}(y_n, t_n | y_1, t_1; y_2, t_2; \ldots; y_{n-1}, t_{n-1}) = \mathcal{P}(y_n, t_n | y_{n-1}, t_{n-1})$$

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Any $\mathcal{P}(y_1, t_1)$ and $\mathcal{P}(y_2, t_2 | y_1, t_1)$ define a Markov process, provided they fulfill:

$$\mathcal{P}(y_3, t_3 | y_1, t_1) = \int \mathrm{d}y_2\, \mathcal{P}(y_3, t_3 | y_2, t_2)\mathcal{P}(y_2, t_2 | y_1, t_1) \qquad \text{(Chapman-Kolmogorov)}$$

$$\mathcal{P}(y_2, t + \Delta t) = \int \mathrm{d}y_1\, \mathcal{P}(y_2, t + \Delta t | y_1, t)\mathcal{P}(y_1, t) \qquad \text{(Evolution equation)}$$

Brownian motion is a Markov process.

Table of Contents

Fokker-Planck equation

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$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x} D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2} D^{(2)}(x,t) \right] \mathcal{P}(x,t).$$

From stochastic processes to Fokker-Planck

- Fokker-Planck is an approximation in linear order

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$$M_n(x, t, \Delta t) := \frac{1}{n!} \langle [x(t + \Delta t) - x(t)]^n \rangle \Big|_{x(t)=x}$$
$$= \frac{1}{n!} \int \mathrm{d}y \, (y - x)^n \mathcal{P}(y, t + \Delta t | x, t)$$

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- Evolution equation (*"we must have come from somewhere"*) and normalization

$$\mathcal{P}(x, t+\Delta t) = \int \mathrm{d}y\,\mathcal{P}(x, t+\Delta t | y, t)\mathcal{P}(y, t) \qquad \text{(evolution)}$$

$$\mathcal{P}(x, t) = \int \mathrm{d}y\,\mathcal{P}(y, t+\Delta t | x, t)\mathcal{P}(x, t) \qquad \text{(normalization)}$$

From stochastic processes to Fokker-Planck

$$\int \mathrm{d}x\, \varphi(x) \cdot \frac{\partial \mathcal{P}(x,t)}{\partial t} \Delta t = \int \mathrm{d}x\, \varphi(x) \underbrace{\left[\mathcal{P}(x, t+\Delta t) - \mathcal{P}(x,t)\right]}_{\text{Taylorexpansion in } \Delta t}$$

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$$= \int \mathrm{d}x \int \mathrm{d}y\, \varphi(y) \underbrace{\mathcal{P}(y, t+\Delta t | x, t)\mathcal{P}(x,t)}_{\text{Evolution equation}} - \int \mathrm{d}x\, \varphi(x) \mathcal{P}(x,t)$$

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$$\int \mathrm{d}x\, \varphi(x) \cdot \frac{\partial \mathcal{P}(x,t)}{\partial t} \Delta t = \int \mathrm{d}x\, \varphi(x) \left[\mathcal{P}(x, t+\Delta t) - \mathcal{P}(x,t)\right]$$

$$= \int \mathrm{d}x \int \mathrm{d}y\, \varphi(y) \mathcal{P}(y, t+\Delta t | x, t) \mathcal{P}(x,t) - \int \mathrm{d}x\, \varphi(x) \mathcal{P}(x,t)$$

$$= \underbrace{\int \mathrm{d}x\, \mathcal{P}(x,t) \int \mathrm{d}y\, [\varphi(y) - \varphi(x)] \mathcal{P}(y, t+\Delta t | x, t)}_{\text{rearrange the terms}}$$

From stochastic processes to Fokker-Planck

$$\begin{aligned}
\int \mathrm{d}x\, \varphi(x) \cdot \frac{\partial \mathcal{P}(x,t)}{\partial t} \Delta t &= \int \mathrm{d}x\, \varphi(x) \left[\mathcal{P}(x, t+\Delta t) - \mathcal{P}(x,t)\right] \\
&= \int \mathrm{d}x \int \mathrm{d}y\, \varphi(y) \mathcal{P}(y, t+\Delta t | x, t) \mathcal{P}(x,t) - \int \mathrm{d}x\, \varphi(x) \mathcal{P}(x,t) \\
&= \int \mathrm{d}x\, \mathcal{P}(x,t) \int \mathrm{d}y\, [\varphi(y) - \varphi(x)] \mathcal{P}(y, t+\Delta t | x, t) \\
&= \int \mathrm{d}x\, \mathcal{P}(x,t) \underbrace{\sum_{n=1}^{\infty} (\partial^n \varphi)(x) \frac{1}{n!} \int \mathrm{d}y\, (y-x)^n \mathcal{P}(y, t+\Delta t | x, t)}_{\text{Taylorexpansion of } \varphi}
\end{aligned}$$

From stochastic processes to Fokker-Planck

$$\int \mathrm{d}x \, \varphi(x) \cdot \frac{\partial \mathcal{P}(x,t)}{\partial t} \Delta t = \int \mathrm{d}x \, \varphi(x) \left[\mathcal{P}(x, t+\Delta t) - \mathcal{P}(x,t) \right]$$

$$= \int \mathrm{d}x \int \mathrm{d}y \, \varphi(y) \mathcal{P}(y, t+\Delta t | x, t) \mathcal{P}(x,t) - \int \mathrm{d}x \, \varphi(x) \mathcal{P}(x,t)$$

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$$= \int \mathrm{d}x \, \mathcal{P}(x,t) \sum_{n=1}^{\infty} (\partial^n \varphi)(x) \underbrace{\frac{1}{n!} \int \mathrm{d}y \, (y-x)^n \mathcal{P}(y, t+\Delta t | x, t)}_{= M_n(x,t,\Delta t)}$$

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\end{aligned}$$

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&= \int \mathrm{d}x \int \mathrm{d}y\, \varphi(y) \mathcal{P}(y, t+\Delta t | x, t) \mathcal{P}(x,t) - \int \mathrm{d}x\, \varphi(x) \mathcal{P}(x,t) \\
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&= \int \mathrm{d}x\, \varphi(x) \cdot \sum_{n=1}^{\infty} (-\partial_x)^n D^{(n)}(x,t) \mathcal{P}(x,t) \Delta t
\end{aligned}$$

From stochastic processes to Fokker-Planck

$$\int \mathrm{d}x\,\varphi(x)\cdot\frac{\partial\mathcal{P}(x,t)}{\partial t}\Delta t = \int \mathrm{d}x\,\varphi(x)\,[\mathcal{P}(x,t+\Delta t)-\mathcal{P}(x,t)]$$

$$= \int \mathrm{d}x \int \mathrm{d}y\,\varphi(y)\mathcal{P}(y,t+\Delta t|x,t)\mathcal{P}(x,t) - \int \mathrm{d}x\,\varphi(x)\mathcal{P}(x,t)$$

$$= \int \mathrm{d}x\,\mathcal{P}(x,t)\int \mathrm{d}y\,[\varphi(y)-\varphi(x)]\mathcal{P}(y,t+\Delta t|x,t)$$

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$$= \int \mathrm{d}x\,\varphi(x)\cdot\sum_{n=1}^{\infty}(-\partial_x)^n D^{(n)}(x,t)\mathcal{P}(x,t)\Delta t$$

From stochastic processes to Fokker-Planck

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = \sum_{n=1}^{\infty} (-\partial_x)^n D^{(n)}(x,t)\mathcal{P}(x,t) \qquad \text{(General Fokker-Planck)}$$

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Continuity equation with probability current density $J(x,t)$.

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Continuity equation with probability current density $J(x,t)$. Assume $J|_{boundary} = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int dx\, \mathcal{P}(x,t) = -\int \mathrm{d}x\, \frac{\partial J(x,t)}{\partial x} = J|_{boundary} = 0$$

Fokker-Planck: Relation to non-equilibrium thermodynamics

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Entropy rate = $\frac{\mathrm{d}S}{\mathrm{d}t}$

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assume $D^{(1)} = F$ (force) and $D^{(2)} = const$.

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$$\propto \int \mathrm{d}x \, \frac{J^2}{\mathcal{P}} - \int \mathrm{d}x \, J \cdot F$$

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$$\begin{aligned}\text{Entropy rate} = \frac{\mathrm{d}S}{\mathrm{d}t} &= -\int \mathrm{d}x \, \frac{\partial}{\partial t} \{\mathcal{P} \log \mathcal{P}\} \\ &= -\int \mathrm{d}x \, \frac{\partial \mathcal{P}}{\partial t}(1 + \log \mathcal{P}) \\ &\propto \int \mathrm{d}x \, \frac{J^2}{\mathcal{P}} - \int \mathrm{d}x \, J \cdot F \\ &= \text{Internal entropy production} - \text{Entropy flux}\end{aligned}$$

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Fokker-Planck: Relation to non-equilibrium thermodynamics

$$
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&= -\int \mathrm{d}x \, \frac{\partial \mathcal{P}}{\partial t} (1 + \log \mathcal{P}) \\
&\propto \int \mathrm{d}x \, \frac{J^2}{\mathcal{P}} - \int \mathrm{d}x \, J \cdot F \\
&= \text{Internal entropy production} - \text{Entropy flux}
\end{aligned}
$$

Entropy rate $\Rightarrow$ Change in probability density $\mathcal{P}$

assume $D^{(1)} = F$ (force) and $D^{(2)} = const$.

Fokker-Planck: Kramers-Moyal expansion coefficients

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = \sum_{n=1}^{\infty} (-\partial_x)^n D^{(n)}(x,t)\mathcal{P}(x,t) \qquad \text{(General Fokker-Planck)}$$

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Kramers-Moyal expansion coefficients $D^{(n)}(x,t)$: Linear approximation of stochastic moments $M_n(x,t)$

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Kramers-Moyal expansion coefficients $D^{(n)}(x,t)$: Linear approximation of stochastic moments $M_n(x,t)$

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But is the physical meaning of these coefficients $D^{(n)}(x,t)$? $\rightarrow$ Langevin equation

Two worlds: Stochastics and physics

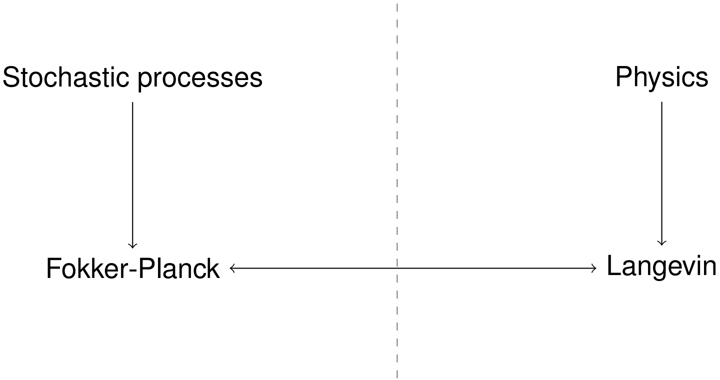
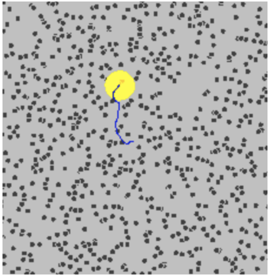


Table of Contents

Langevin equation: Newton's equation with random force ζ



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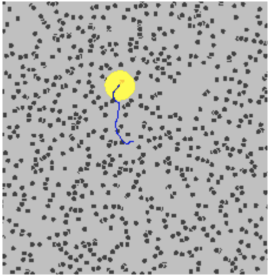
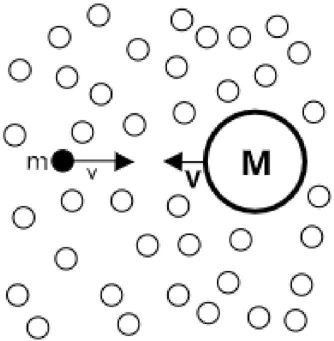


$$\underbrace{m\ddot{x}}_{\text{inertia}} = \underbrace{-\gamma\dot{x}}_{\text{friction}} + \underbrace{F(x)}_{\text{external forces}} + \underbrace{\zeta(t)}_{\text{random force}}$$

Let’s motivate Langevin’s equation microscopically (1)



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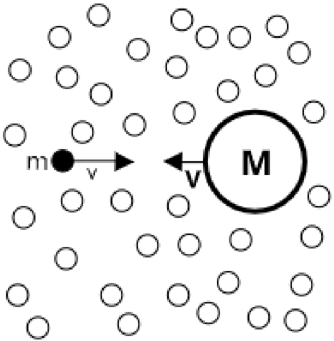

Velocity after one elastic collision:

$$
\begin{aligned}
V' &= \frac{M-m}{M+m} V + \frac{2m}{M+m} v \\
&= \left(1 - \frac{2m}{M}\right) V + \frac{2m}{M} v + \mathcal{O}\left(\frac{m^2}{M^2}\right)
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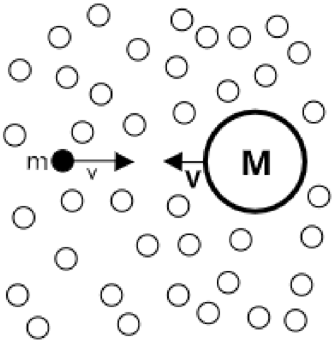

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Change in momentum for one collision:

$$\Delta P = 2mv - 2mV$$

Let's motivate Langevin's equation microscopically (2)

Change of momentum for N collisions during time span Δt with avg collision rate $n = \frac{N}{\Delta t}$:

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Emergent macroscopic properties from microscopic effects:

- *Drag coefficient*: scales with number of solvent particles $\gamma = 2mn$
- *Random force*: sum of many microscopic collisions $\zeta = 2\sum_{i=1}^{n\Delta t} \frac{mv_i}{\Delta t}$ ($\rightarrow$ Gaussian)

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Next: Two concrete examples on Brownian motion

Overdamped limit, step 1: Langevin equation

Langevin equation with no force term $F(x)$ and overdamped ($m \ll \gamma$)

$$\cancel{m\ddot{x}} = -\gamma\dot{x} + \cancel{F(x)} + \zeta(t) \quad \longrightarrow \quad \gamma\dot{x} = \zeta(t)$$

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Two assumptions necessary to compute these moments.

Overdamped limit, step 2: Moments M_1, M_2

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Take expectation for first moment

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Square and take expectation to get second moment

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Overdamped limit, step 3: Fokker-Planck

In summary we found:

$$M_1(x_t, t, \Delta t) = 0, \qquad M_2(x_t, t, \Delta t) = \frac{\Gamma}{2\gamma^2}\Delta t = D^{(2)}(x, t)\Delta t$$

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Fokker-Planck equation for this problem:

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[D^{(1)}(x,t) - \frac{\partial}{\partial x}D^{(2)}(x,t)\right]\mathcal{P}(x,t) = \frac{\Gamma}{2\gamma^2}\frac{\partial^2 \mathcal{P}(x,t)}{\partial x^2}$$

This is a *diffusion equation*.

Next Langevin model: Velocity diffusion

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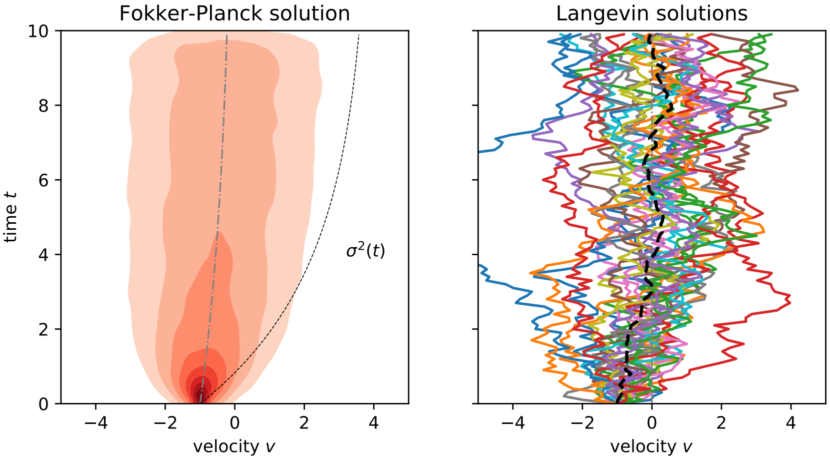
3. Fokker-Planck equation: a *convection-diffusion equation*

$$\frac{\partial \mathcal{P}(v,t)}{\partial t} = \frac{1}{m}\left[\gamma \frac{\partial}{\partial v} v + \frac{\Gamma}{2m} \frac{\partial^2}{\partial v^2}\right] \mathcal{P}(v,t).$$

Velocity diffusion: what do solutions look like?



Stochastic Langevin simulations: $\frac{v(t+\Delta t)-v(t)}{\Delta t} = -\frac{\gamma}{m}v(t) + \frac{1}{m}\zeta(t)$

```
m,gamma,Gamma = 1, 0.15, 1.2          # Langevin parameters
n_timesteps, endtime = 100, 10        # discretization parameters
dt    = endtime / n_timesteps
v     = np.zeros(n_timesteps)         # initialize discrete solution
v[0]  = -1                            # initial velocity

for t in range(1, n_timesteps):
    noise = np.random.normal(0, np.sqrt(Gamma))
    v[t]  = v[t-1]
    v[t] -= dt * gamma  / m * v[t-1]
    v[t] += np.sqrt(dt) / m * noise
```

Velocity diffusion: stationary solution

Stationary Fokker-Planck solution satisfies $0 = \frac{\partial \mathcal{P}(v,t)}{\partial t} = \frac{1}{m}\left[\gamma \frac{\partial}{\partial v} v + \frac{\Gamma}{2m}\frac{\partial^2}{\partial v^2}\right]\mathcal{P}(v)$, thus

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$$\Gamma = 2\gamma k_B T$$

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$$\mathcal{P}(v) \propto \exp\left(-\frac{m\gamma}{\Gamma}v^2\right) = \exp\left(-\frac{m}{2k_BT}v^2\right) \equiv \mathcal{P}_{\text{Boltzmann}}$$

Einstein relation: second moment Γ (fluctuation) related to kinetic energy (dissipation)

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Velocity diffusion: non-stationary solution

Non-stationary solution for initial condition $\mathcal{P}(v, 0) = \delta(v - v_0)$:

(non-stationary)

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$$\mathcal{P}(v, t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp\left(-\frac{(v - \langle v(t) \rangle)^2}{2\sigma^2(t)}\right) \qquad \text{(non-stationary)}$$

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with time-dependent variance $\sigma^2(t) = \frac{\Gamma}{2m\gamma}(1 - e^{-2\gamma t})$ and mean $\langle v(t)\rangle = v(0)e^{-\gamma t/m}$.

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$$\mathcal{P}(v, t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp\left(-\frac{(v - \langle v(t)\rangle)^2}{2\sigma^2(t)}\right) \qquad \text{(non-stationary)}$$

with time-dependent variance $\sigma^2(t) = \frac{\Gamma}{2m\gamma}(1 - e^{-2\gamma t})$ and mean $\langle v(t)\rangle = v(0)e^{-\gamma t/m}$.

Limit $t \to \infty$ recovers stationary solution

$$\mathcal{P}(v) \propto \exp\left(-\frac{m\gamma}{\Gamma} v^2\right) \equiv \mathcal{P}_{\text{Boltzmann}} \qquad \text{(stationary)}$$

Not limited to Brownian motion: Stochastic harmonic oscillator

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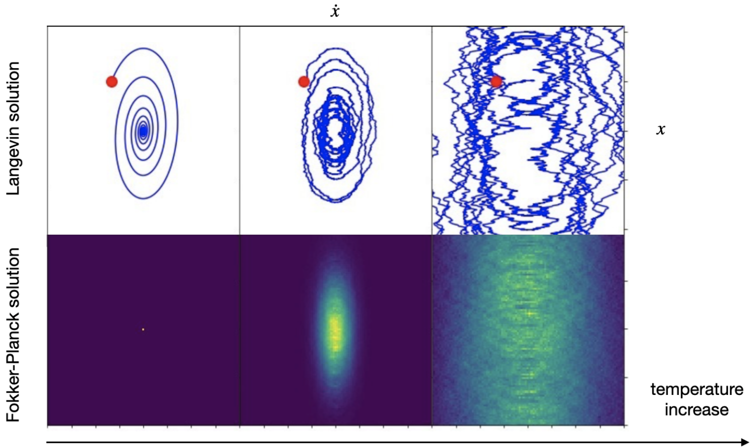

What have we learnt?

- We could find the Fokker-Planck equations for two physical systems

$$\frac{\partial \mathcal{P}(v,t)}{\partial t} = \frac{\Gamma}{2\gamma^2}\frac{\partial^2 \mathcal{P}(x,t)}{\partial x^2} \quad \text{and} \quad \frac{\partial \mathcal{P}(v,t)}{\partial t} = \frac{1}{m}\left[\gamma\frac{\partial}{\partial v}v + \frac{\Gamma}{2m}\frac{\partial^2}{\partial v^2}\right]\mathcal{P}(v,t)$$

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- In the limit $t \to \infty$ we reach thermal equilibrium

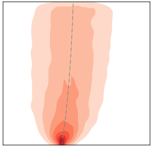
$$\mathcal{P}(v,t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}}\exp\left(-\frac{(v - \langle v(t)\rangle)^2}{2\sigma^2(t)}\right) \xrightarrow{t\to\infty} \mathcal{P}_{\text{Boltzmann}}$$

Two worlds: Stochastics and physics

Stochastic processes → Fokker-Planck

Physics → Langevin

Fokker-Planck ↔ Langevin



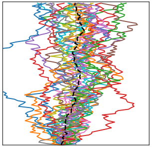


Table of Contents

Stock market

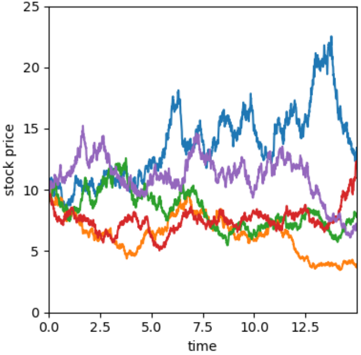


Figure: Stock price evolution for different random fluctuations.

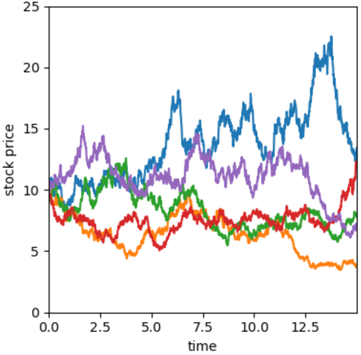
Stock price S has fluctuating growth rate

$$\frac{dS}{dt} = (\mu + \zeta(t))S \quad \text{and} \quad \langle \zeta(t)\zeta(t') \rangle = \sigma^2 \delta(t - t')$$

μ: drift

σ: volatility

Stock market



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Find the probability distribution $\mathcal{P}(S, t)$

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$$\frac{ds}{dt} = \left(\mu - \frac{\sigma^2}{2}\right) + \zeta(t) \rightarrow \frac{\partial \mathcal{P}(s,t)}{\partial t} = \frac{\partial}{\partial s}\left[\left(\mu - \frac{\sigma^2}{2}\right)\mathcal{P}(s,t) - \frac{\sigma^2}{2}\frac{\partial \mathcal{P}(s,t)}{\partial s}\right]$$

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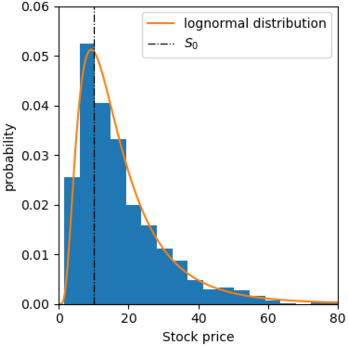

Figure: Histogram for 1000 instances of stock price evolution compared to the expected probability distribution.

Initial condition: $\mathcal{P}(S, 0) = \delta(S - S_0)$

Solutions for $s = \log(S)$ and S

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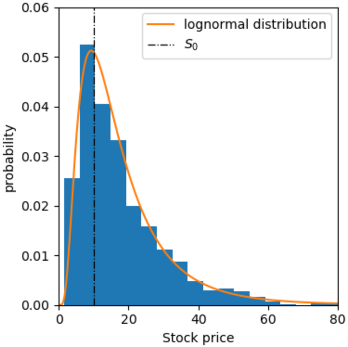

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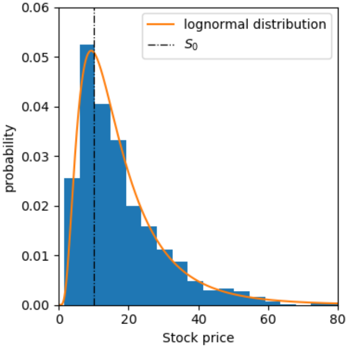

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Stock market: Properties of the solution

The mean of stock price probability distribution is

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Note:
If we used standard derivation rules in the change of variables the growth rate of $\langle S \rangle$ would be $\mu + \sigma^2/2$.

Spreading of a virus

We extend our stock market model by the term $(N_{max} - x)$

$$\dot{x} = (r_0 + \zeta)x \rightarrow \dot{x} = (r_0 + \zeta)x$$

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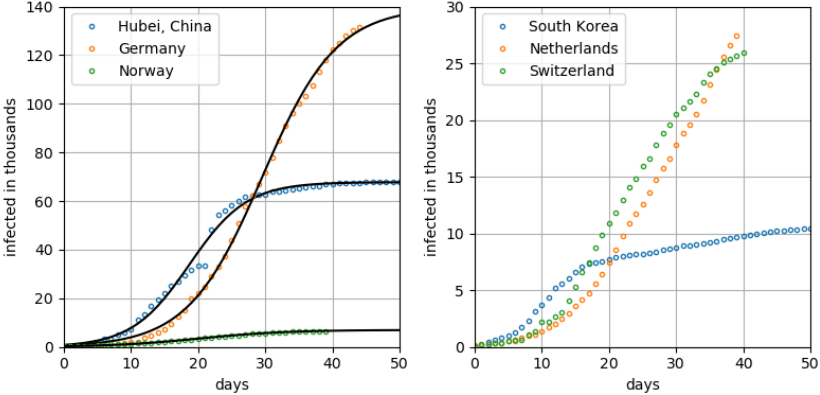
For parameter estimation we use the solution of the logistic differential equation

$$x(t) = \frac{N_{max}x(0)}{x(0) + e^{-r_0 N_{max} t}(N_{max} - x(0))}$$

Parameter estimation



Parameter estimation

Table: Estimates for the parameters N_{max} and r_0. The standard deviation σ can not be estimated from a single trajectory.

country	N_{max}	r_0	σ
Hubei, China	68000	$3.46 \cdot 10^{-6}$	?
Germany	140000	$1.29 \cdot 10^{-6}$	?
Norway	7000	$21.89 \cdot 10^{-6}$	?

Fokker-Planck equation

The Fokker-Planck equation for Langevin equation with multiplicative noise $\dot{x} = F(x) + b(x)\zeta$ reads

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[F(x)\mathcal{P}(x,t) - b(x)\frac{\partial}{\partial x}\left[b(x)\mathcal{P}(x,t)\right]\right]$$

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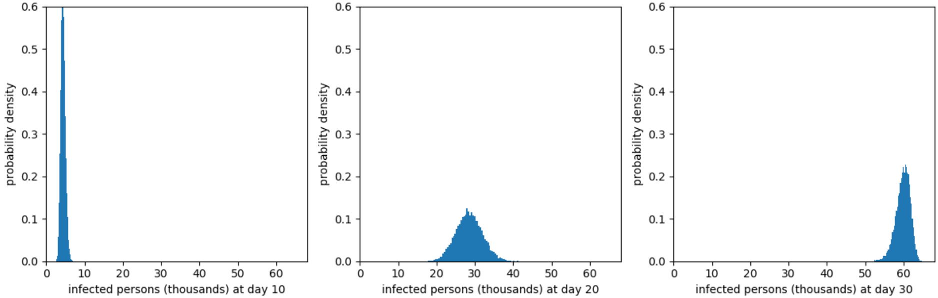
thus the Fokker-Planck equation for the logistic model is

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[r_0 x(N_{max} - x)\mathcal{P}(x,t) - \sigma^2 x(N_{max} - x)\frac{\partial}{\partial x}\left[x(N_{max} - x)\mathcal{P}(x,t)\right]\right]$$

Numerical solution



Two stationary solutions:

$\delta(x - N_{max})$ for positive infection rate and $\delta(x)$ for negative infection rate

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- It is hard to estimate parameters in a stochastic model without an underlying microscopic theory.

Fundamentals: Probability theory, stochastic processes & Markov processes

Fokker-Planck equation

Langevin equation

Applications: Stock price & Virus spreading

Path integral formulation

Conceptual introduction to Keldysh formalism

Why path integral formulation?

So far: A *classical* theory to understand *stochastic* physics.

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- Close analogy between classical stochastic physics and quantum mechanics

$$\text{quantum mechanics} \quad \sim \quad \text{classical stochastic physics}$$

- Derivation of classical stochastic physics from QFT in classical limit $\hbar \to 0$

$$\text{non-equilibrium QFT} \quad \longrightarrow \quad \text{classical stochastic physics}$$

Comparison quantum mechanics and stochastics

Notion	Quantum mechanics	Stochastics (MSR)
Central object	State $\Psi(x, t)$	Probability $\mathcal{P}(x, t)$
State space	location x, momentum p	?
Propagator	$\langle x, t \vert x_0, t_0 \rangle$	?
Hamiltonian	$H(x, p)$	?
Action	$\int \mathrm{d}t\, L(x, \dot{x})$	?
Time evolution	Schroedinger eq	Fokker-Planck eq

Characterizations of $\zeta(t)$: Moments vs. Probability distribution

So far:

Characterize probability distribution of random force $\zeta(t)$, a *random variable* at any time t, implicitly through its moments

$$\langle \zeta(t) \rangle = 0 \quad \text{and} \quad \langle \zeta(t)\zeta(t') \rangle = \Gamma\delta(t - t').$$

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How to reformulate stochastic physics through path integrals

Partition function:

Z = sum entire sample space

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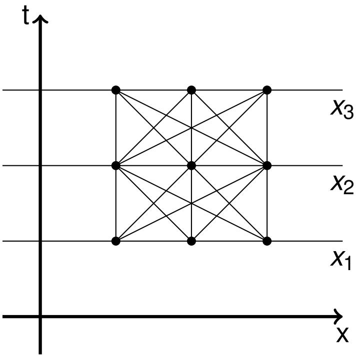
What is $\mathcal{P}[\zeta(t)]$ and how to think of $\int \mathcal{D}x\, \delta(L[x, \dot{x}] - \zeta)$?

Path integrals: Summing over all paths



We need to make sure $\sum_{x(t)} \mathcal{P}[x(t)] = 1$. But how do we sum over all trajectories?

Discretise time $t_1 < t_2 < ... < t_n$ and approximate the path by the points $x_i = x(t_i)$.

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_n \, \mathcal{P}(x_1, ..., x_n)$$
$$\overset{n\to\infty}{\longrightarrow} \int \mathcal{D}x \, \mathcal{P}[x(t)]$$

Let's try this out for Langevin equation.

Path integrals: Langevin equation

Recall the Langevin equation ($\gamma = 1$), $\dot{x} - F(x) = \zeta(t)$, and define

$$L[x, \dot{x}] := \dot{x} - F(x) = \zeta(t) \quad \xrightarrow{\text{discretize}} \quad L_n := \frac{x_n - x_{n-1}}{\Delta t} - F(x_n) = \zeta_n$$

Given a random force, Langevin trajectories are deterministic $\rightarrow$ δ-functions

$$1 = \int_{\mathbb{R}^n} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \prod_{k=1}^{n} \Delta t \delta(L_k - \zeta_k) \quad \longrightarrow \quad 1 = \int \mathcal{D}x \, \delta(L[x(t), \dot{x}(t)] - \zeta(t))$$

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Use Fourier representation of delta function $\delta(x) = \int \mathcal{D}p\, e^{\int \mathrm{d}t\, ipx}$

$$\int \mathcal{D}x\, \delta(L[x(t), \dot{x}(t)] - \zeta(t)) = \int \mathcal{D}x \mathcal{D}p \exp\left(\int dt\, 2ip(L[x, \dot{x}] - \zeta) \right)$$

Probability of a random force trajectory $\mathcal{P}(t)$

Probability for a random trajectory (without proof)

$$\mathcal{P}[\zeta(t)] = \mathit{normalization} \cdot \exp\left(-\int_{-\infty}^{+\infty} \mathrm{d}t\, \frac{\zeta^2(t)}{2\Gamma}\right)$$

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Correlators defined as

$$\langle \zeta(t_1)\zeta(t_2)...\zeta(t_n)\rangle := \int \mathcal{D}\zeta\, \zeta(t_1)\zeta(t_2)...\zeta(t_n)\mathcal{P}[\zeta]$$

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Consistency check: Above probability distribution yields previously postulated correlators

$$\langle \zeta(t)\rangle = 0 \quad \text{and} \quad \langle \zeta(t)\zeta(t')\rangle = \Gamma\delta(t - t')$$

Putting it all together: Partition function

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$$= \int \mathcal{D}x\mathcal{D}p\mathcal{D}\zeta \ \exp\left(-\int dt \left(\frac{1}{2\Gamma}\zeta^2 + 2ip\zeta - 2ipL[x,\dot{x}]\right)\right)$$

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Martin-Siggia-Rose (MSR) action and Hamiltonian

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(Lagrangian)

(Hamiltonian)

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Note: p is *not* a momentum, but an auxiliary variable.

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\begin{aligned}
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&= \int \mathcal{D}x\mathcal{D}p \, \exp\left(-2i\int dt \, pL[x,\dot{x}] + i\Gamma p^2\right) && \longleftarrow \text{Action} \\
&= \int \mathcal{D}x \, \exp\left(-\frac{1}{2\Gamma}\int dt \, (L[x,\dot{x}])^2\right) && \longleftarrow \text{Propagator}
\end{aligned}
$$

Propagators

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Recall: Probabilities (propagators) are restricted partition functions:

$$\mathcal{P}(x, t | x_0, t_0) = \frac{1}{Z} \cdot \text{noise trajectories } \zeta(t) \cdot \text{Langevin trajectories } x(t) \text{ that start in } x_0$$
$$\text{at time } t_0 \text{ and end in } x \text{ at time } t, \text{ given noise } \zeta(t)$$

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Related to the propagator in quantum mechanics by a *Wick rotation*

$$\langle x,t|x_0,t_0\rangle = \int_{x_0}^{x} \mathcal{D}x \, \exp\left(\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}t \, L(x,\dot{x})\right)$$

Derivation of Fokker-Planck equation

The idea is to use the evolution equation for a small time step Δt

$$\begin{aligned}\mathcal{P}(x, t + \Delta t) &= \int \mathrm{d}y\, \mathcal{P}(x, t + \Delta t | y, t)\mathcal{P}(y, t) \\ &= \int \mathrm{d}(\delta x)\, \mathcal{P}(x, t + \Delta t | x - \delta x, t)\mathcal{P}(x - \delta x, t).\end{aligned}$$

From the path integral formulation we know the propagator

$$\mathcal{P}(x, t + \Delta t | x - \delta x, t) = \frac{1}{\sqrt{2\pi\Delta t}} \exp\left(-\frac{1}{2\Gamma}\Delta t \left(\frac{\delta x}{\Delta t} - F(x - \delta x)\right)^2\right)$$

Fokker-Planck equation

Some Taylor expansion in δx and Δt later we get the Fokker-Planck equation

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[F(x)\mathcal{P}(x,t) - \frac{\Gamma}{2}\frac{\partial \mathcal{P}(x,t)}{\partial x}\right].$$

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The Fokker-Planck equation can be transformed into a *Schrödinger like* form

$$\frac{i}{2}\partial_t \mathcal{P}(x,t) = H[x,p]\mathcal{P}(x,t)$$

with Hamiltonian $H(x,p) = pF(x) - i\Gamma p^2$ by identifying the variable $p \to -\frac{i}{2}\partial_x$.

Comparison quantum mechanics and stochastics

Notion	Quantum mechanics	Stochastics (MSR)
Central object	State $\Psi(x,t)$	Probability $\mathcal{P}(x,t)$
State space	location x, momentum p	location x, *auxiliary* variable p
Propagator	$\langle x,t \vert x_0,t_0\rangle$	$\mathcal{P}(x,t \vert x_0,t_0)$
Hamiltonian	$H(x,p)$	$H[x,p] = pF(x) - i\Gamma p^2$
Action	$\int \mathrm{d}t\, L(x,\dot{x})$	$\int \mathrm{d}t\, L^2[x,\dot{x}]$
Time evolution	Schroedinger eq	Fokker-Planck eq

Fundamentals: Probability theory, stochastic processes & Markov processes

Fokker-Planck equation

Langevin equation

Applications: Stock price & Virus spreading

Path integral formulation

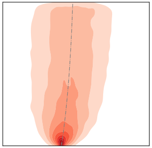
Conceptual introduction to Keldysh formalism

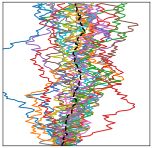
Two worlds: stochastics and physics

Stochastic Processes → Fokker-Planck

Physics → Langevin

Fokker-Planck ↔ Langevin



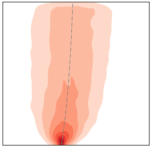


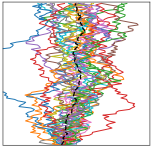
Two worlds: stochastics and physics

Stochastic Processes → Fokker-Planck

QFT → Langevin

Fokker-Planck ⟷ Langevin





Keldish formalism: Why and how?

Why? Derivation of Langevin equation from non-equilibrium quantum field theory

How? Summary of following slides, a *conceptual* derivation

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1. Define QFT partition function Z

Keldish formalism: Why and how?

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1. Define QFT partition function Z
2. QFT partition function $\rightarrow$ path integral formulation

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How? Summary of following slides, a *conceptual* derivation

1. Define QFT partition function Z
2. QFT partition function $\rightarrow$ path integral formulation
3. Path integral partition function Z $\rightarrow$ Keldysh action S for single particle

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4. Single particle action $S \rightarrow$ Brownian many particle action S_{Keldysh}
5. Classical limit of many particle Keldysh action $S_{\text{Keldysh}} \rightarrow$ Langevin equation (inverse of derivation of MSR action from Langevin equation)

Recap on quantum (statistical) mechanics

Von-Neumann equation: time evolution of mixed state $\rho(t)$

$$\partial_t \rho(t) = -i\hbar \left[H(t), \rho(t) \right]$$

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Solutions of von Neumann equation expressed through *Evolution operator* $\mathcal{U}_{t,t'}$ that transforms state at time t' to state at time t

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$$\langle \mathcal{O} \rangle (t) = \frac{\mathrm{Tr}\left\{\mathcal{O}\rho(t)\right\}}{\mathrm{Tr}\left\{\rho(t)\right\}} =$$

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Keldysh: Time contour and partition function

Define partition function in analogy to observable average

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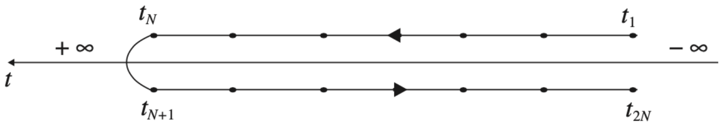
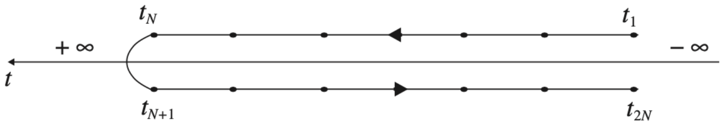
$$Z := \frac{\mathrm{Tr}\left\{\mathcal{U}_{-\infty,\infty}\mathcal{U}_{\infty,-\infty}\rho(-\infty)\right\}}{\mathrm{Tr}\left\{\rho(-\infty)\right\}}$$

Call $\mathcal{U}_{-\infty,\infty}\mathcal{U}_{\infty,-\infty}$ the *time contour* $\mathcal{C}$:



Keldysh: From partition function Z to action S

Concrete: Hamiltonian for particle in potential V: $H = \frac{p^2}{2m} + V(x)$



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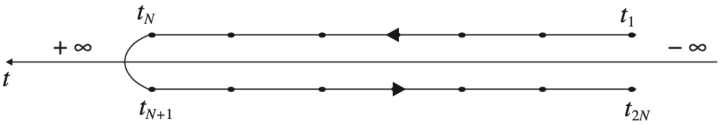

Discretize time contour to obtain path integral representation of Z with action $S[x, \dot{x}]$

$$Z = \frac{1}{\mathrm{Tr}\{\rho(-\infty)\}} \int \mathrm{d}x_1 \ldots \mathrm{d}x_{2n} \prod_{k=1}^{2n-1} \langle x_{k+1} | \mathcal{U}_{t_{k+1},t_k} | x_k \rangle \langle x_1 | \rho(-\infty) | x_{2n} \rangle$$

$$\longrightarrow \frac{1}{\mathrm{Tr}\{\rho(-\infty)\}} \int \mathcal{D}x \, \exp \underbrace{\left(\frac{i}{\hbar} \int_{\mathcal{C}} \mathrm{d}t \, \frac{1}{2} \dot{x}^2 - V(x) \right)}_{:= \frac{i}{\hbar} S[x, \dot{x}]}$$

Keldysh: Keldysh action for Brownian particle

Single particle: Action for particle in potential $V(x)$ is $S[x,\dot{x}] = \int_{\mathcal{C}} \mathrm{d}t\, \frac{1}{2}\dot{x}^2 - V(x)$.
Split x into forward/backward contour parts x^+, x^- (s.t. "$\int_{\mathcal{C}} \to \int_{\mathbb{R}}$") and do a Keldysh rotation $2x^{cl} = x^+ + x^-, 2x^q = x^+ - x^-$:

$$S[x,\dot{x}] = \int_{\mathcal{C}} \mathrm{d}t\, \frac{1}{2}\dot{x}^2 - V(x) \quad \longrightarrow \quad S[x^{cl}, x^q] = -\int_{\mathbb{R}} \mathrm{d}t\, 2x^q(\ddot{x}^{cl} + V'(x^{cl}))$$

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Many particles: Total action for Brownian particle in potential $V(X)$ in contact with bath of harmonic oscillators $S_{\text{Keldysh}} = S_{\text{particle}} + S_{\text{bath}} + S_{\text{interaction}}$.

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Many particles: Total action for Brownian particle in potential $V(X)$ in contact with bath of harmonic oscillators $S_{\text{Keldysh}} = S_{\text{particle}} + S_{\text{bath}} + S_{\text{interaction}}$. In classical limit

$$S_{\text{Keldysh}} \longrightarrow \int \mathrm{d}t\, \left[-2x^q[\ddot{x}^{cl} + \gamma\dot{x}^{cl} + V'(x^{cl})] + 4i\gamma T \cdot (x^q)^2\right] \quad \text{as } \hbar \to 0.$$

Keldysh: Derivation of Langevin equation

Inverse steps of the MSR transformation: introduce auxiliary field ζ using Hubbard-Stratonovich transformation $\exp(-(x^q)^2/2a) \propto \int \mathrm{d}\zeta \, \exp(-\frac{\zeta^2}{2a} - i x^q \zeta)$:

$$Z = \int \mathcal{D}x^{cl}\mathcal{D}x^q \, e^{iS}$$

Keldysh: Derivation of Langevin equation

Inverse steps of the MSR transformation: introduce auxiliary field ζ using Hubbard-Stratonovich transformation $\exp(-(x^q)^2/2a) \propto \int \mathrm{d}\zeta \, \exp(-\frac{\zeta^2}{2a} - ix^q\zeta)$:

$$Z = \int \mathcal{D}x^{cl}\mathcal{D}x^q \, e^{iS}$$
$$= \int \mathcal{D}\zeta\mathcal{D}x^{cl} \, e^{-\frac{1}{4\gamma T}\int \mathrm{d}t\,\zeta^2} \int \mathcal{D}x^q \exp\left\{-2i\int \mathrm{d}t\, x^q(\ddot{X}^{cl} + \gamma\dot{x}^{cl} + V'(x^{cl}) - \zeta)\right\}$$

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\begin{aligned}
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&= \int \mathcal{D}\zeta\mathcal{D}x^{cl} \, e^{-\frac{1}{4\gamma T}\int \mathrm{d}t\,\zeta^2} \, \delta\left(\ddot{x}^{cl} + \gamma\dot{x}^{cl} + V'(x^{cl}) - \zeta\right)
\end{aligned}
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\end{aligned}
$$

Only trajectories $x^{cl}(t)$ that satisfy *Langevin equation* enter path integral

$$
\ddot{x}^{cl} = -\gamma\dot{x}^{cl} - V'(x^{cl}) + \zeta(t) \qquad \longleftrightarrow \qquad \ddot{x} = -\gamma\dot{x} + F(x) + \zeta(t)
$$

Summary

- Fokker-Planck equation describes the evolution in time of a stochastic system

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[F(x) - \frac{\partial}{\partial x}\frac{\Gamma}{2}\right]\mathcal{P}(x,t)$$

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- The path integral formulation formally relates stochastic process to quantum mechanics. The MSR action takes an important role in this analogy

$$S[x,p] = \int \mathrm{d}t\, pL[x,\dot{x}] + i\Gamma p^2 = \int \mathrm{d}t\, p\dot{x} - pF(x) + i\Gamma p^2$$

Two worlds: Stochastics and physics

Stochastic Processes → Fokker-Planck

Physics → Langevin

Fokker-Planck ↔ Langevin