Langevin, Fokker-Planck, Path Integrals & Keldysh

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1 Probability theory, stochastic processes & Markov processes

We want to model stochastic physical systems. Our main example of interest will be that of *Brownian motion*, i.e. a particle in a fluid that experiences random forces from the surrounding molecules. Since this influence is random the trajectories of the particles are not deterministic anymore and we have to resort to a probabilistic description. For this purpose we introduce the notion of stochastic processes.

First, let's briefly recall some probability basics. A *random variable* X is a function $\Omega \to \mathbb{R}^n$ from the state space Ω into the reals. The state space Ω is the set of all possibilities. In the case of the Brownian particle this would be the set of all possible trajectories $x(t)$. An example for a random variable is the evaluation of the trajectories at a certain time t_1. Usually we are not interested in the precise form of the state space Ω and the random variable X. Instead we characterize the random variable X by its *probability density function* (pdf) $\mathcal{P}_A(x_1, ..., x_n)$. A probability density function $\mathcal{P}(x_1, ..., x_n)$ captures the probability that the random variable has a value within the volume element $dx_1...dx_n$. This could for example be the probability to find a Brownian particle at position (x_1, x_2, x_3). Of course, the probabilities need to sum to one

$$\int \mathrm{d}x_1 ... \mathrm{d}x_n \, \mathcal{P}(x_1, ..., x_n) = 1.$$

The only probability distribution essential for this talk is the *normal distribution* with pdf

$$\mathcal{P}(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

A normally distributed random variable, $Z \sim \mathcal{N}(\mu, \sigma^2)$, has mean μ and variance σ^2.

For a random variable $X : \Omega \to \mathbb{R}$, we may compute statistical *moments*

$$\langle X^n \rangle := \int \mathrm{d}x \, x^n \mathcal{P}(x).$$

The first moment $\langle X \rangle$ is called the *mean*, the second (centralized) moment $\langle X^2 \rangle - \langle X \rangle^2$ the *variance*. For two random variables X and Y we define the probability $\mathcal{P}(x, t)$ of finding X at x and Y at y. Note that $\mathcal{P}(x, y)$ is also the probability density of the random variable $\omega \in \Omega \mapsto \binom{X(\omega)}{Y(\omega)}$. With this definition we can calculate the *correlation* $\langle XY \rangle$ between to random variables

$$\langle XY \rangle = \int \mathrm{d}x \mathrm{d}y \, xy \cdot \mathcal{P}(x, y),$$

A *stochastic process* is a set of random variables $y(t)$ where t denotes time. These random variables $y(t)$ could for example be the position or velocity of a Brownian particle. Any

random variable is characterized by the probability density $\mathcal{P}(y_1, t_1)$ to have $y(t_1) = y_1$. The transition probability $\mathcal{P}(y_2, t_2|y_1, t_1)$ describes the probability that $y(t_2) = y_2$ under the condition that (earlier in time) $y(t_1) = y_1$. We can generalize this to transition probabilities with several conditions $\mathcal{P}(y_n, t_n|y_1, t_1; y_2, t_2; \ldots; y_{n-1}, t_{n-1})$.

The molecules surrounding a Brownian particle move much faster than the particle itself. This means that the time scale on which the molecules reach an equilibrium state after hitting the Brownian particle is much smaller than the time scale important for the movement of the Brownian. Therefore the random forces acting on the particle at a time t_2 does not depend on where the particle was at $t_1 < t_2$, i.e the particle has no memory. This property leads us to a special class of stochastic processes, the Markov process.

A *Markov processes* satisfies for $t_1 < t_2 < \ldots < t_n$

$$\mathcal{P}(y_n, t_n|y_1, t_1; y_2, t_2; \ldots; y_{n-1}, t_{n-1}) = \mathcal{P}(y_n, t_n|y_{n-1}, t_{n-1}), \qquad \text{(Markov property)}$$

i.e. the transition probability from y_{n-1} at time t_{n-1} to the new state y_n at t_n depends only the state y_{n-1}, but not on all previous states $y_1, \ldots, y_{n-2}$. Hence, Markov processes are fully determined by $\mathcal{P}(y_1, t_1)$ and $\mathcal{P}(y_2, t_2|y_1, t_1)$. A sufficient requirement for these probability densities to define a Markov process is the following set of equations

$$\mathcal{P}(y_3, t_3|y_1, t_1) = \int \mathrm{d}y_2\, \mathcal{P}(y_3, t_3|y_2, t_2)\mathcal{P}(y_2, t_2|y_1, t_1), \qquad \text{(Chapman-Kolmogorov)}$$

$$\mathcal{P}(y_2, t + \Delta t) = \int \mathrm{d}y_1\, \mathcal{P}(y_2, t + \Delta t|y_1, t)\mathcal{P}(y_1, t). \qquad \text{(Evolution equation)}$$

Roughly speaking, the Chapman-Kolmogorov equation requires that transition probabilities can be arbitrarily subdivided, while the evolution equation states that a stochastic process must have come from somewhere back in time.

An example for a Markov process is the so called *Wiener process*, defined by the two probability distributions

$$\mathcal{P}(y_2, t_2|y_1, t_1) = \frac{1}{\sqrt{2\pi|t_2 - t_1|}} \exp\left(-\frac{(y_2 - y_1)^2}{2|t_2 - t_1|}\right),$$

$$\mathcal{P}(y, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right).$$

The Wiener process describes the behavior of a particle under the influence of a random force as we will see below.

2 Fokker-Planck equation

These few preliminaries already allow us to derive the famous Fokker-Planck equation, a PDE that governs the time evolution of the probability density $\mathcal{P}(x, t)$ to find a particle at position x at time t, following the derivations from Essler and Krüger [1, 2].

2.1 Derivation

We are going to expand the probability density $\mathcal{P}(x,t)$ in the moments of the transition probability

$$M_n(x,t,\Delta t) := \frac{1}{n!}\langle [x(t+\Delta t)-x(t)]^n\rangle = \frac{1}{n!}\int \mathrm{d}y\,(y-x)^n \mathcal{P}(y,t+\Delta t|x,t) \qquad (1)$$

by Taylor expanding in time. It is helpful to first work out the weak version of the PDE using a test function $\varphi(x)$ equipped with all the "nice" mathematical properties (smooth and goes to zero sufficiently fast) we are going to need. Up to linear order, we may write

$$\int \mathrm{d}y\,\varphi(y)\cdot\frac{\partial \mathcal{P}(y,t)}{\partial t}\Delta t = \int \mathrm{d}y\,\varphi(y)\left[\mathcal{P}(y,t+\Delta t)-\mathcal{P}(y,t)\right].$$

Using the evolution equation and by relabelling the integration variables

$$\int \mathrm{d}x\,\varphi(x)\cdot\frac{\partial \mathcal{P}(x,t)}{\partial t}\Delta t = \int \mathrm{d}y\int \mathrm{d}x\,\varphi(y)\mathcal{P}(y,t+\Delta t|x,t)\mathcal{P}(x,t) - \int \mathrm{d}x\,\varphi(x)\mathcal{P}(x,t).$$

From the normalization requirement $1=\int \mathrm{d}y\,\mathcal{P}(y,t+\Delta t|x,t)$, Taylor expansion, and partial integration we find

$$\begin{aligned}
\int \mathrm{d}x\,\varphi(x)\cdot\frac{\partial \mathcal{P}(x,t)}{\partial t}\Delta t &= \int \mathrm{d}x\,\mathcal{P}(x,t)\int \mathrm{d}y\,[\varphi(y)-\varphi(x)]\mathcal{P}(y,t+\Delta t|x,t)\\
&= \int \mathrm{d}x\,\mathcal{P}(x,t)\sum_{n=1}^{\infty}(\partial^n\varphi)(x)\underbrace{\frac{1}{n!}\int \mathrm{d}y\,(y-x)^n\mathcal{P}(y,t+\Delta t|x,t)}_{M_n(x,t,\Delta t)}\\
&= \int \mathrm{d}x\,\varphi(x)\sum_{n=1}^{\infty}(-\partial_x)^n M_n(x,t,\Delta t)\mathcal{P}(x,t).
\end{aligned}$$

Let's abandon the test functions and resort back to the strong version of the equation by noticing that the above equality must hold for all test functions $\varphi(x)$,

$$\frac{\partial \mathcal{P}(x,t)}{\partial t}\Delta t = \sum_{n=1}^{\infty}(-\partial_x)^n M_n(x,t,\Delta t)\mathcal{P}(x,t).$$

Next, we may expand the moments M_n up to linear order $M_n(x,t,\Delta t) = D^{(n)}(x,t)\Delta t + O(\Delta t^2)$. By taking the limit $\Delta t\to 0$ we obtain a generalized Fokker-Planck equation

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = \sum_{n=1}^{\infty}(-\partial_x)^n\left[D^{(n)}(x,t)\mathcal{P}(x,t)\right].$$

Usually, only the first two terms are kept (or non-zero), with which we finally arrive at the textbook version of the *Fokker-Planck equation*[1]

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x} D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2} D^{(2)}(x,t) \right] \mathcal{P}(x,t). \tag{2}$$

This equation is commonly supplemented with the initial condition $\mathcal{P}(x,t) = \delta(x - x_0)$. Of course, one may generalize the equation to a larger state space $\mathbf{x}$

$$\frac{\partial \mathcal{P}(\mathbf{x},t)}{\partial t} = \sum_{|\alpha| \geq 1} D^{(n)}(\mathbf{x}, t, \Delta t)(-\partial)^{\alpha} \mathcal{P}(\mathbf{x}, t).$$

Note that the Fokker-Planck equation remains an approximation, after all. Still, this approximation is useful and fulfills an important constraint: it conserves probability. We may check this by pulling out one differentiation in (2) to obtain a continuity-like equation

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = -\frac{\partial}{\partial x} \left[D^{(1)}(x,t) - \frac{\partial}{\partial x} D^{(2)}(x,t) \right] \mathcal{P}(x,t) = -\frac{\partial J(x,t)}{\partial x}$$

with probability current

$$J(x,t) = D^{(1)}(x,t)\mathcal{P}(x,t) - \frac{\partial}{\partial x} \left[D^{(2)} \mathcal{P}(x,t) \right]. \tag{3}$$

If we assume this probability current to vanish on the system boundaries, the total probability is conserved.

2.2 Relation to non-equilibrium thermodynamics

We will not cover any thermodynamics in following chapters, but we will try to provide some intuition on the relation between the Fokker-Planck equation and thermodynamics in this section. As in statistical mechanics, the link between stochastics and thermodynamics can be established through entropy, which is defined for a probability distribution as

$$S = -\int \mathrm{d}x \, \mathcal{P} \log \mathcal{P}.$$

The Fokker-Planck equation describes time derivatives of the probability density $\mathcal{P}$ and so we are interested in the time derivative of entropy, the entropy rate

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\int \mathrm{d}x \, \frac{\partial}{\partial t} \{\mathrm{P} \log \mathrm{P}\} = -\int \mathrm{d}x \, \frac{\partial \mathcal{P}}{\partial t} (1 + \log \mathcal{P}).$$

[1] Note that in this notation the derivatives act on all parts further to the right of the derivative, e.g. the first derivative acts on $\frac{\partial}{\partial x}\left[D^{(1)}(x,t)\mathcal{P}(x,t)\right]$.

Here we may halt and obtain our first insight: a constant in time probability density $\mathcal{P}$ implies zero entropy change, or the other way round

$$\text{Time-varying total entropy} \implies \text{Time-varying probability density } \mathcal{P}.$$

The entropy rate for irreversible processes splits into two contributions: the entropy flux across the system's boundaries due to e.g. heat or mass flow and the internal irreversible entropy production. Imagine for example a cell as a system that takes up sugar molecules across its membrane, an entropy flux, and breaks down the molecules inside, an irreversible process. How does our stochastic version of the entropy rate relate to the thermodynamics?

To gain some intuition about the connection, we first take advantage of the beauty that the Fokker-Planck equation allows us to exchange the time derivative with a state space derivative of the probability density $\mathcal{P}$, using the probability current J

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \int \mathrm{d}x \, \frac{\partial J}{\partial x}(1 + \log \mathcal{P}) = -\int \mathrm{d}x \, J \frac{\partial \log \mathcal{P}}{\partial x},$$

where we used partial integration and the assumption that the probability current J vanishes on the boundary. Next, we rearrange the definition of the probability current (3) to express $\frac{\partial \mathcal{P}}{\partial x}$ in terms of $J, D^{(1)}, D^{(2)}$,

$$\frac{\partial \mathcal{P}}{\partial x} = \frac{1}{D^{(2)}} \left[\left(D^{(1)} - \frac{\partial D^{(2)}}{\partial x} \right) \mathcal{P} - J \right] \propto F\mathcal{P} - J,$$

where in the second equality we made the assumption that $D^{(1)}$ is proportional to a force F and $D^{(2)}$ is constant. These are reasonable assumptions that hold for all examples in this document. We find that the entropy rate splits into two terms

$$\frac{\mathrm{d}S}{\mathrm{d}t} \propto \int \mathrm{d}x \, \frac{J^2}{\mathcal{P}} - \int \mathrm{d}x \, J \cdot F.$$

The first term is positive definite, while the second term has the form of a flux. Thus we may attribute the first term to the internal entropy production due to irreversibility of processes within the system, while the second term corresponds to an entropy flux across the system boundaries.

3 Langevin equation

Our derivation of the Fokker-Planck equation was purely mathematical. In this section we are going to discuss the physics behind Fokker-Planck, following Tong and Essler [3, 1]. Our starting point will be the so called *Langevin equation*, which is a model for the stochastic trajectory of a particle subject to random fluctuations. The idea of Langevin's equation is

to extend Newton's equations by a random force $\zeta(t)$, which is mathematically a random variable. This leads to the following stochastic differential equation, the general Langevin equation

$$m\ddot{\mathbf{x}} = -\gamma\dot{x} + F(\mathbf{x}) + \zeta(t). \qquad \text{(Langevin equation)}$$

Usually, we take the random force $\zeta(t)$ to be Gaussian, justified by the central limit theorem which roughly states that a large sum of independent random forces will be normally distributed. Furthermore, we assume our random force at different times $\zeta(t'), \zeta(t)$ to be uncorrelated

$$\langle \zeta(t)\zeta(t') \rangle = \Gamma\delta(t - t'). \tag{4}$$

In the following, we are going to discuss the Langevin equation for the classical example of Brownian particles that perform a random walk as a result of constantly bumping into water molecules, the sum of which is what the particle feels as a random force $\zeta(t)$. This will provide us with physical expressions for the cryptic $D^{(n)}$ factors in the Fokker-Planck equation (2).

3.1 Microscopic motivation

First, we are going to provide a microscopic motivation for the Langevin equation of a Brownian particle, as presented by de Grooth *et. al.* [4]. Consider the *Brownian setting*: a bath containing many small balls, the solute molecules, of mass m and initial velocity v and one large ball, the Brownian particle, of mass M and initial velocity V. After an elastic collision between the Brownian particle and one solute molecule, the Brownian particle's new velocity V' is given by

$$V' = \frac{M-m}{M+m}V + \frac{2m}{M+m}v = \left(1 - \frac{2m}{M}\right)V + \frac{2m}{M}v + \mathcal{O}\left(\frac{m^2}{M^2}\right).$$

Here, we have assumed the Brownian particle to be much heavier than the solute molecules, $m/M \ll 1$. This leads to a change in momentum for the Brownian particle of $\Delta P = 2mv - 2mV$. Now consider a small timescale Δt, during which the velocity of the Brownian particle $V(t)$ does not change much. Within Δt, on average $N = n\Delta t$ collisions take place. The Brownian particle's total change in momentum for N collisions during Δt is then given by

$$\Delta P = 2m\sum_{i=1}^{N} v_i - 2m\sum_{i=1}^{N} V_i = 2m\sum_{i=1}^{n\Delta t} v_i - 2mV(t)n\Delta t.$$

In the limit $\Delta t \to 0$ we obtain the equation of motion for the Brownian particle

$$M\dot{V} = -\gamma V + \zeta, \qquad \text{with} \qquad \gamma = 2mn, \quad \zeta = \frac{1}{\Delta t}\sum_{i=1}^{n\Delta t} 2mv_i.$$

By comparing to the Langevin equation we postulated before, we see in this motivation how the random force ζ emerges as the macroscopic effect of the sum of a large number of microscopic collisions between the smaller and the Brownian particles.

3.2 Diffusion in the overdamped limit

Now consider the simplest case of a small Brownian particle in a solvent subject to random forces $\zeta(t)$. This particle experiences a drag force $-\gamma\dot{x}$, something like a friction force. We assume the solvent to be so viscous that $\gamma/m \gg 1$, which allows us to neglect the inertia term $m\ddot{x}$. In this so called overdamped limit the Langevin equation reads

$$\gamma\dot{x} = \zeta(t).$$

The integration of the above equation leads to the solution as a function of a stochastic integral

$$x(t) = x(0) + \frac{1}{\gamma}\int_0^t \mathrm{d}t'\,\zeta(t').$$

We are interested in the statistical properties of this solution instead of exact values.. For this reason, we compute the first and second centralized moments of $x(t)$. Recall that averages and integrals are linear operations, so we can pull the average into the integration and use our assumptions on the first and second moment of the random force ζ

$$\langle x(t) - x(0)\rangle = \frac{1}{\gamma}\int_0^t \mathrm{d}t'\,\underbrace{\langle\zeta(t')\rangle}_{=0} = 0,$$

$$\langle (x(t) - x(0))^2\rangle = \frac{1}{\gamma^2}\int_0^t\int_0^t \mathrm{d}t'\mathrm{d}t''\,\underbrace{\langle\zeta(t')\zeta(t'')\rangle}_{=\Gamma\delta(t'-t'')} = \frac{\Gamma}{\gamma^2}t.$$

Note that we can always extract the mean of a random force into the deterministic external force part F of the Langevin equation and by that rescale ζ to have zero mean, so that our assumption of vanishing mean random force $\langle\zeta(t)\rangle = 0$ is not a limitation. Also note that the $\sqrt{t}$ dependence of the root-mean-square distance is characteristic for diffusive motion, $\langle x^2\rangle \sim t$, in contrast to ballistic motion with $x^2 \sim t^2$.

Using equation (1), we can now relate these moments to the the factors $M_1(x,t,\Delta t)$ inside the Fokker-Planck equation (2)

$$M_1(x,t,\Delta t) = \langle x(t+\Delta t) - x(t)\rangle = 0$$
$$M_2(x,t,\Delta t) = \langle [x(t+\Delta t) - x(t)]^2\rangle = \frac{\Gamma}{\gamma^2}\Delta t$$

and hence $D^{(1)}(x,t) = 0, D^{(2)}(x,t) = \frac{\Gamma}{\gamma^2}$. Thus we obtain our first physically motivated Fokker-Planck equation

$$\frac{\partial\mathcal{P}(x,t)}{\partial t} = \frac{\Gamma}{2\gamma^2}\frac{\partial^2\mathcal{P}(x,t)}{\partial x^2}. \tag{5}$$

The Fokker-Planck equation for this problem simplifies to the *diffusion equation*, how cool is that?

3.3 Velocity diffusion

Next, we include the inertia term $m\ddot{x}$ and relabel $\dot{x} \to v$. The resulting Langevin's equation is

$$m\dot{v} + \gamma v = \zeta(t). \tag{6}$$

Let's also tackle this Langevin equation. Using $\frac{\mathrm{d}}{\mathrm{d}t}\left(v(t)e^{\gamma t/m}\right) = (\dot{v} + \frac{\gamma}{m}v)e^{\gamma t/m} = \frac{1}{m}\zeta(t)e^{\gamma t}$, we obtain the integral solution

$$v(t) = v(0)e^{-\gamma t/m} + \frac{1}{m}\int_0^t \mathrm{d}t'\, \zeta(t')e^{-\gamma(t-t')/m}.$$

Again, we may compute the moments

$$\langle v(t)\rangle = v(0)e^{-\gamma t/m} + \frac{1}{m}\int_0^t \mathrm{d}t'\, \underbrace{\langle\zeta(t')\rangle}_{=0} e^{-\gamma(t-t')/m} = v(0)e^{-\gamma t/m},$$

$$\langle v^2(t)\rangle = v^2(0)e^{-2\gamma t/m} + \frac{\Gamma}{2m\gamma}(1 - e^{-2\gamma t/m}) = \langle v(t)\rangle^2 + \sigma^2(t),$$

with time-dependent variance $\sigma^2(t) = \frac{\Gamma}{2m\gamma}(1 - e^{-2\gamma t/m})$. Similarly, if a little more cumbersome, we obtain the Fokker-Planck equation's moment factors M_n for this setting

$$\begin{aligned}
M_1(v,t,\Delta t) &= \langle v(t+\Delta t) - v(t)\rangle = -\frac{\gamma}{m}v(0)e^{-\gamma t/m}\Delta t + O(\Delta t^2),\\
M_2(v,t,\Delta t) &= \langle [v(t+\Delta t) - v(t)]^2\rangle = \frac{\Gamma}{m^2}\Delta t + O(\Delta t^2)\\
M_n(v,t,\Delta t) &= \langle [v(t+\Delta t) - v(t)]^n\rangle = O(\Delta t^2), \qquad \text{for } n \geq 3,
\end{aligned}$$

and from this, the Fokker-Planck equation for this setting, a *convection-diffusion equation*

$$\frac{\partial \mathcal{P}(v,t)}{\partial t} = \frac{1}{m}\left[\gamma\frac{\partial}{\partial v}v + \frac{\Gamma}{2m}\frac{\partial^2}{\partial v^2}\right]\mathcal{P}(v,t). \tag{7}$$

The above equation extends equation (5) with a convective term proportional to $\partial_v(v\mathcal{P})$.

3.4 Properties of Fokker-Planck solutions

Let's briefly discuss some properties of solutions to the Fokker-Planck equation (7). Simple solution techniques such as separation do not work for Fokker-Planck equations, see section A.3 in the appendix for a detailed explanation. This is why Green's functions are commonly

used to solve the Fokker-Planck equation, but in general, Green's functions are a useful tools to compute solution of any partial differential equation. *Green's functions* describe the response of system to a disturbance. In this particular case of Fokker-Planck equation we are interested how the probability distribution $\mathcal{P}(v,t)$ responds, if the probability that the Brownian particle has velocity v' at time t' is one.

Mathematically speaking, this means that the Green's function $G(v,v';t,t')$ fulfills the Fokker-Planck equation

$$\frac{\partial G(v,v';t,t')}{\partial t} + \frac{\partial}{\partial v}\left[D^{(1)} - \frac{\partial^2}{\partial v^2}D^{(2)}(v)\right] G(v,v';t,t') = \delta(v-v')\delta(t-t').$$

We can calculate solution for an arbitrary initial condition $\mathcal{P}(v',t')$ by convolution

$$\mathcal{P}(v,t) = \int \mathrm{d}v'\, G(v,v';t,t')\mathcal{P}(v',t').$$

This is the evolution equation we have seen for Markov processes. Therefore may identify $\mathcal{P}(v,t|v',t') = G(v,v';t,t')$.

The Green's function can be found by Fourier transforming the Fokker-Planck equation and solving the corresponding equation in momentum space. For the overdamped limit we find the Green's function

$$G(x,x';t,t') = \frac{1}{\sqrt{2\pi(t-t')\Gamma}} \exp\left(-\frac{(x-x')^2}{2(t-t')\Gamma}\right). \tag{8}$$

By the identification of the Green's function with the transition probability we see that the diffusion in the overdamped limit is a Wiener process.

For the diffusion equation (7), the solution is a Gaussian function

$$\mathcal{P}(v,t|v_0,0) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp\left(-\frac{(v-\langle v(t)\rangle)^2}{2\sigma^2(t)}\right) \tag{9}$$

with time dependent mean $\langle v(t)\rangle = v_0 e^{-\gamma t/m}$ and variance $\sigma^2(t) = \frac{\Gamma}{2m\gamma}(1-e^{-2\gamma t})$. Figure 1 shows a plot of this probability distribution as a function of time and velocity as well as 25 possible Langevin trajectories sampled from the Langevin equation (6). Here, by "possible Langevin trajectory" we mean that these trajectories solve the Langevin equation given *some* realization of the random force stochastic process ζ.

Initially at $t = 0$, all probability mass is concentrated around the initial value $v_0 = -1$, this corresponds to vanishing variance $\sigma^2(t)$. Soon after $t > 0$, the probability distribution broadens, but probability mass and trajectories move towards 0: even though we set off the particle with an initial velocity, it will after some time loose this drift and converge to a vanishing velocity, on average, due to friction $-\gamma v$ and random forces. In the equilibrium limit, the mean velocity vanishes $\langle v(t)\rangle \to 0$, but the fluctuations increase to a constant,

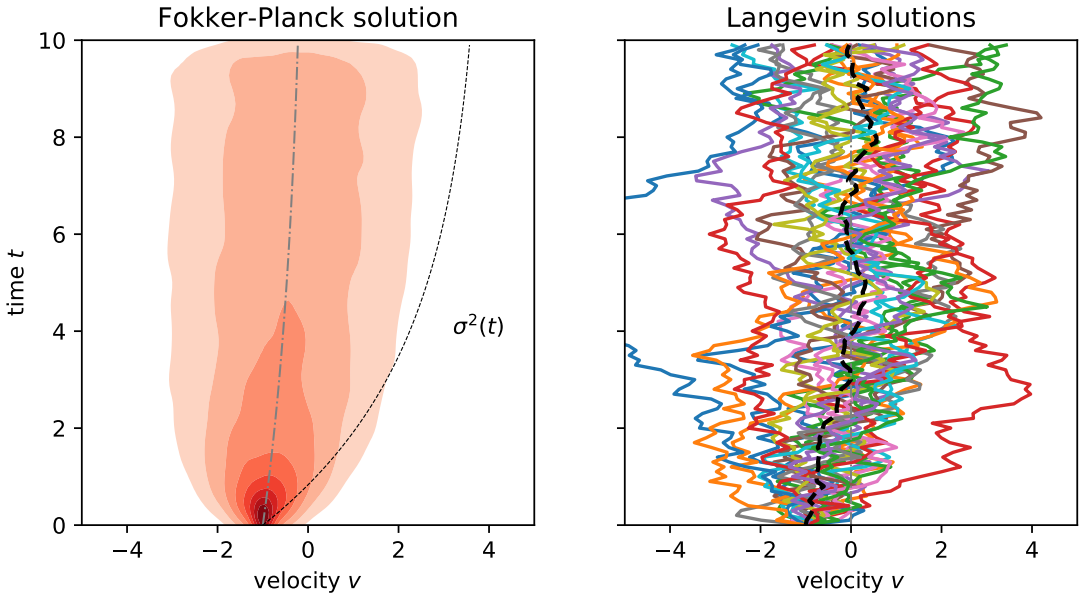


Figure 1: Time evolution of Fokker-Planck and Langevin solutions for initial condition $v_0 = -1$. The Fokker-Planck solution is a probability distribution $\mathcal{P}(v,t)$ and its plot was obtained by sampling 1000 velocity samples per timepoint from the solution (9). Darker red corresponds to higher probability mass. Solutions to the Langevin equation (6) are individual trajectories $v_\zeta(t)$ depending on the stochastic force $\zeta(t)$. The plot shows 25 Langevin solutions, the bold black dashed line is the mean of all trajectory.

$\langle v^2(t)\rangle = \sigma^2(t) \to \frac{\Gamma}{2m\gamma} = \frac{1}{m}\gamma D$ as $t \to \infty$, where we have redefined the correlation $\Gamma = 2\gamma^2 D$. In thermodynamics, we have learned about the equipartition theorem which states that $\frac{1}{2}m\langle v^2\rangle \sim \frac{1}{2}k_B T$. Taken together, this leads to the famous *Einstein-Stokes relation*

$$D = \frac{k_B T}{\gamma}, \tag{10}$$

in which D takes the role of a diffusion constant. This is a specific occurrence of a *fluctuation dissipation theorem*, which states that dissipation produces fluctuations and the other way round. Here, the drag force γ feels like friction to the particles, which dissipates kinetic energy (dissipation). However, Einstein-Stokes tells us that this dissipation leads to diffusive motion of the particles (a fluctuation).

The Einstein-Stokes relations also allows for a consistency check of the above solution (9) in the equilibrium limit $t \to \infty$. In our case, the solution (9) converges to the *Maxwell-*

Boltzmann distribution, as expected

$$\begin{aligned}\mathcal{P}(v,t) &\to \sqrt{\frac{m\gamma}{\pi\Gamma}}\exp\left(-\frac{m\gamma v^2}{\Gamma}\right)\\ &= \sqrt{\frac{m}{2\pi k_B T}}\exp\left(-\frac{mv^2}{2k_B T}\right) = \mathcal{P}_{\text{Boltzmann}}(v), \qquad \text{as } t\to\infty.\end{aligned}$$

4 Examples and application

Stochastic models are not limited to Brownian motion and physics. Such models are for example used in financial mathematics, from which we present a simple stock market model. In addition we propose a simple stochastic model of virus spreading, because of the recent events regarding the Corona virus.

4.1 Stock market & multiplicative noise Langevin equations

The proposed model for the stock market and some of the results in this section were taken from Asiri [5]. In the proposed model the stock price fullfills the linear differential equation

$$\dot{S} = (\mu + \zeta)S \tag{11}$$

The parameter μ is called the drift and has the meaning of an average growth rate. The variable ζ is a random Gaussian noise with the same type of correlation as the random force in the Langevin equation

$$\langle\zeta(t)\zeta(t')\rangle = \sigma^2\delta(t-t').$$

with volatility σ. The volatility determines how sensitive the stock price is to the random fluctuations.

We want to find the probability density $\mathcal{P}(S,t)$ of the stock price, but in contrast to our previous Langevin equations the noise ζ is multiplied by the stock price S, which is called *multiplicative noise*. In order to find a differential equation without multiplicative noise we make a change of variables $s = \log(S)$. This change of variables is a bit problematic, because we can not use the standard derivation rules. Due to the diffusive motion of the random fluctuations the change in S is only $\sigma \mathrm{d}t^{1/2}$. Together with the change originating from the drift μ we get

$$\mathrm{d}S = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}t^{1/2}. \tag{12}$$

A more detailed explanation for why the change in S is proportional to $\mathrm{d}t^{1/2}$ can be found in appendix A.1. The change for $\mathrm{d}s$ is

$$\mathrm{d}s = \log(S+\mathrm{d}S) - \log(S) = \log(1+\mu\mathrm{d}t + \sigma\mathrm{d}t^{1/2}) \tag{13}$$

$$= \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma\mathrm{d}t^{1/2} + \mathcal{O}(\mathrm{d}t^{3/2}). \tag{14}$$

Note the appearance of the additional term $-\sigma^2/2$, compared with the standard derivation rules. In the limit $\mathrm{d}t \to 0$ equation (14) becomes a differential equation for s

$$\dot{s} = \frac{\partial \log(S)}{\partial t} = \mu - \frac{\sigma^2}{2} + \zeta.$$

where we identified the term $\sigma \mathrm{d}t^{1/2}$ with the random fluctuations ζ.

Now it is straightforward to find the moments for this stochastic process and thereby the Fokker-Planck equation for the probability density of s

$$\frac{\partial \mathcal{P}(s,t)}{\partial t} = -\frac{\partial}{\partial s}\left[\left(\mu - \frac{\sigma^2}{2}\right)\mathcal{P}(s,t) - \frac{\sigma^2}{2}\frac{\partial \mathcal{P}(s,t)}{\partial s}\right].$$

In order to find the Green's function we make a second change of variable $u = s - \left(\mu - \frac{\sigma^2}{2}\right)t$

$$\frac{\partial \mathcal{P}(u,t)}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 \mathcal{P}(u,t)}{\partial u^2}.$$

This is the diffusion equation with the Green's function given in equation (8). Thereby, we can deduce the Green's function for the Fokker-Planck equation of s

$$\mathcal{P}(s,t) = \frac{1}{\sqrt{2\pi t}\sigma}\exp\left(-\frac{\left(s - \left(\mu - \frac{\sigma^2}{2}\right)t - s_0\right)^2}{2\sigma^2 t}\right),$$

which means the random variable $s(t)$ is normally distributed for any time t. Since $S = e^s$ the random variable S has to be log-normally distributed

$$\mathcal{P}(S,t) = \frac{1}{\sqrt{2\pi t}\sigma S}\exp\left(-\frac{\left(\log(S) - \left(\mu - \frac{\sigma^2}{2}\right)t - \log(S_0)\right)^2}{2\sigma^2 t}\right). \tag{15}$$

The log-normal distribution is plotted in figure 2. It agrees well with the numerical solution obtained by solving the differential equation (11). To solve this differential equation the same numerical method as for the Langevin equation was used.

The expectation value $\langle S \rangle = S_0 e^{\mu t}$ grows exponentially, which is the solution of the differential equation (11) without random fluctuations. This result is only obtained, because of the additional term $\sigma^2/2$ we got from the change of variables. With standard derivation rules the growth rate of $\langle S \rangle$ would be $\mu + \sigma^2/2$, which justifies the way the Langevin equation of $s = \log(S)$ was derived.

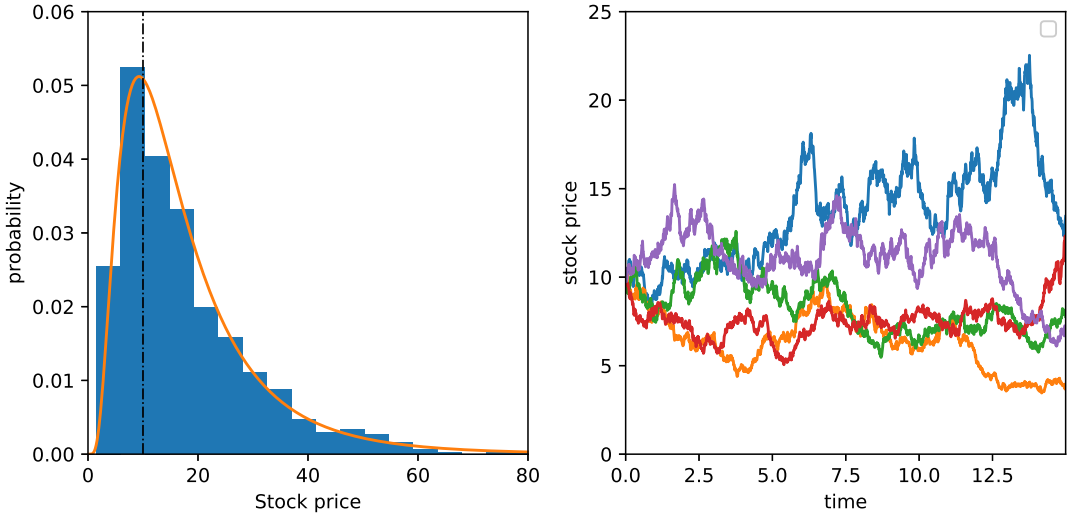


Figure 2: The left picture shows the histogram for 1000 trajectories of the stock price with initial price of 10 (black dotted line). The orange line is the expected log-normal distribution (15) from theory. The right picture shows five instances of the stock price evolution.

4.2 Spreading of a virus

In this subsection we propose a simple stochastic model of virus spreading. We want to model the number of infected persons x. A first simple idea is that we have an infection rate r with a noise $r = r_0 + \zeta$. This leads to the same model as for stock market

$$\dot{x} = (r_0 + \zeta)x$$

The limit of this model are obvious, because the mean of the probability distribution grows exponentially as we have seen for the stock price. To get a bounded curve we multiply an additional term $(x_{\max} - x)$

$$\dot{x} = (r_0 + \zeta)x(x_{\max} - x), \tag{16}$$

where $x_{\max}$ is the limit of infected persons. Without the random fluctuations ζ this is the logistic differential equation. The solution of the deterministic logistic differential equation is

$$x(t) = \frac{x_{\max}x(0)}{x(0) + e^{-rx_{\max}t}(x_{\max} - x(0))} \tag{17}$$

Fitting this solution to the data shown in figure 3 allows us to find estimates for r_0 and $x_{\max}$ shown in table 1. The standard deviation σ of the random fluctuations could not be estimated from the single trajectory of infected persons. The value of $x_{\max}$ varies a lot, because it depends on the population size. But there are also differences in the infection rate r_0, which could be due to different precautions measurements.

country	$x_{\max}$	r_0	σ
Hubei, China	68000	$3.46 \cdot 10^{-6}$	?
Germany	140000	$1.29 \cdot 10^{-6}$	?
Norway	7000	$21.89 \cdot 10^{-6}$	?

Table 1: Estimated parameters from different countries. With these numbers one can estimated the numbers of infected persons with the solution of the logistic differential equation (17). The solution for the model with random fluctuations is shown in 4 based on the estimated parameters from Hubei.

Furthermore, for most of the countries the logistic curve could not be fitted well to the data. In the right plot of figure three examples of such countries are shown. For the Netherlands and Switzerland the curve becomes almost linear after 20 days, which can not described well within the logistic model. One reason could be that in reality the infection rate r_0 and the limit $x_{\max}$ change in time, because of the precaution measurements. This can also explain the kink in the South Korean curve. In conclusion this simple model has its limitation, when it is applied to actual data. Nevertheless the estimated parameters are useful to have the right scale for our model.

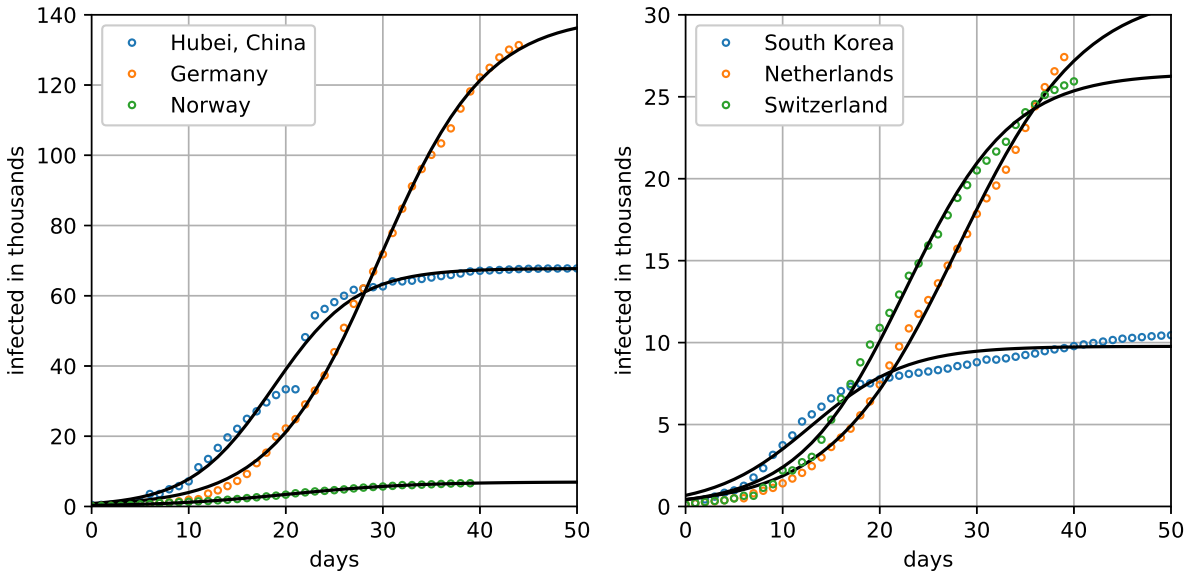


Figure 3: The data of the Corona pandemic is taken from the Johns Hopkins university [6]. On the left plot the data from three different regions is fitted by equation (17) with floating parameter $r, x_{\max}, x(0)$. The right plot shows three examples where the data could not be fitted as well by the logistic model as for the other three examples. We set day 0 to be the date, when the number of confirmed cases exceeded 100.

In a next step we want to find out how the probability distribution evolves in time for the logistic model. The Fokker-Planck equation for this problem is [2]

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[r_0 x(x_{\max}-x)\mathcal{P}(x,t) - \sigma^2 x(x_{\max}-x)\frac{\partial}{\partial x}\left[x(x_{\max}-x)\mathcal{P}(x,t)\right]\right]$$

Since no analytical solution for this PDE could be found, a numerical approach was used. The differential equation (16) was solved numerically for different instances of the random fluctuations. From these solution we could estimate the probability distribution. The results are shown in figure 4. The evolution can be divided into three phases. First, the probability distribution is sharp, because at beginning the number of infected people is known. Due to the random fluctuations the distribution starts broadening and in the

[2]Following Kamenev [7] we get for a general Langevin equation with multiplicative noise

$$\dot{x} = F(x) + b(x)\zeta$$

the corresponding Fokker-Planck equation

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[F(x)\mathcal{P}(x,t) - b(x)\frac{\partial}{\partial x}\left[b(x)\mathcal{P}(x,t)\right]\right]$$

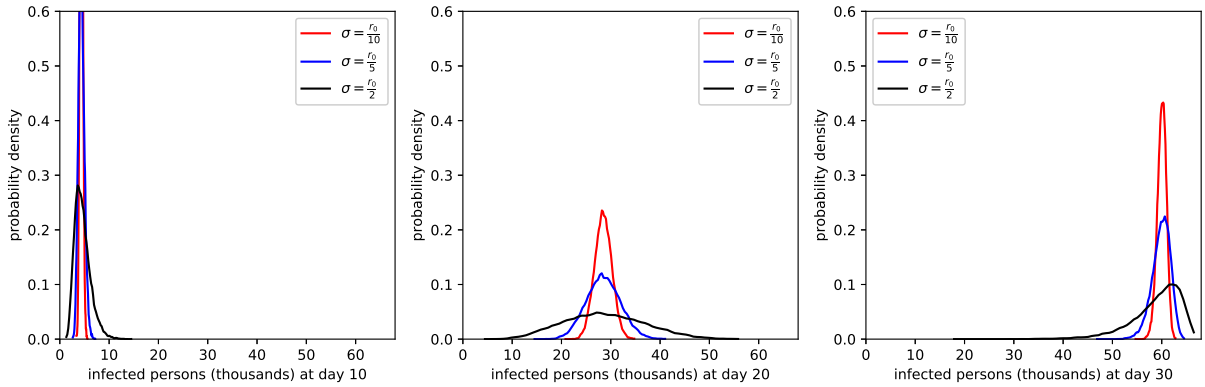


Figure 4: Evolution of the probability distribution in time for different values of σ. The probability distribution is sampled by 20000 trajectories. For r_0 and $x_{\max}$ the estimation from Hubei, China shown in table 1 are used.

middle phase of the spreading it gets harder to predict the exact amount of infected persons. The maximum value of infected person is bounded at $x_{\max}$, thereby all trajectories will be close to this maximum value at some point. In this phase the probability distribution gets sharper again. In the limit $t \to \infty$ we get the stationary solution $\mathcal{P}(x,t) = \delta(x - x_{\max})$. There is another stationary solution $\mathcal{P}(x,t) = \delta(x)$. But this stationary solution is only reached in the limit $t \to \infty$ for a negative the infection rate r_0. Furthermore the solutions for the different values of σ show that the variance of the probability distribution depends on the standard deviation σ of the random fluctuations. On the other hand the mean of the probability distribution does not depend on σ.

5 Path integral formulation

So far, we have specified moments of the random noise ζ probability distribution. An alternative characterization of ζ aims to directly specify the probability $\mathcal{P}[\zeta(t)]$. Since the random force $\zeta(t)$ is a function of time, its corresponding probability must be a functional of ζ and its normalization condition then reads

$$\int \mathcal{D}\zeta\, \mathcal{P}[\zeta(t)] = \sum_{\text{all trajectories } \zeta(t)} \mathcal{P}[\zeta(t)] = 1,$$

i.e. the sum over the probability $\mathcal{P}[\zeta(t)]$ for all possible random noise functionals $\zeta(t)$ must be unity. This naturally leads to the notion of functional integrals, the *path integrals*.

This section will not teach us a lot new about Fokker-Planck theory, but serves mostly technical purposes by introducing notions like path integrals of Langevin equations or Green's functions in a formalism analogous to quantum mechanics that will make it easier

for us to understand the relation of the so far discussed classical stochastic theory to non-equilibrium quantum field theory in the Keldysh formalism. Still, we will learn about the close relation between stochastic and quantum mechanical descriptions.

5.1 Probability distributions over Gaussian stochastic processes

First, we briefly need to discuss Gaussian processes. We would like to find the probability $\mathcal{P}[\zeta(t)]$ for the Gaussian random force we have so far worked with. To reproduce the two-point correlator $\langle\zeta(t)\zeta(t')\rangle = \Gamma\delta(t-t')$ we postulated earlier, cf. equation (4), an educated guess for the probability is

$$\mathcal{P}[\zeta(t)] = \exp\left(-\int_{-\infty}^{+\infty} \mathrm{d}t\, \frac{\zeta^2(t)}{2\Gamma}\right), \tag{18}$$

normalized such that the path integral over all possible noises $\zeta(t)$ is unity, $\int \mathcal{D}\zeta\, \mathcal{P}[\zeta(t)] = 1$. Within this path integral formalism, correlations are defined as

$$\langle\zeta(t_1)\zeta(t_2)...\zeta(t_n)\rangle := \int \mathcal{D}\zeta\, \zeta(t_1)\zeta(t_2)...\zeta(t_n)\mathcal{P}[\zeta].$$

A common tool to calculate the correlators of the noise $\langle\zeta(t_1)\zeta(t_2)\rangle$ is the so called generating function $Z[J(t)]$. The arguments of Z are arbitrary functions $J(t)$. This means $Z[J(t)]$ defines a functional

$$Z[J(t)] = \int \mathcal{D}\zeta\, \mathcal{P}[\zeta(t)] e^{\int \mathrm{d}t\, J(t)\cdot\zeta(t)},$$

from which the correlators can be computed through the functional derivative [3]

$$\langle\zeta(t_1)\rangle = \left.\frac{\delta Z}{\delta J(t_1)}\right|_{J=0},$$
$$\langle\zeta(t_1)\zeta(t_2)\rangle = \left.\frac{\delta Z}{\delta J(t_1)\delta J(t_2)}\right|_{J=0}.$$

All we have to do now is to compute the generating function and its functional derivatives for our probability distribution $\mathcal{P}[\zeta]$. By Gaussian integration we get

$$Z[J] = \int \mathcal{D}\zeta\, \exp\left(-\int_{-\infty}^{\infty} \mathrm{d}t\, \left[\frac{\zeta(t)^2}{2\Gamma} - J(t)\zeta(t)\right]\right) = \exp\left(\int_{-\infty}^{\infty} \frac{\Gamma}{2} J(t)^2\right).$$

[3] Let $\phi(x)$ be a function and $\mathcal{F}[\phi]$ a functional. The functional derivative $f(x) = \frac{\delta\mathcal{F}}{\delta\phi}|_{\phi(x)}$ is the function $f(x)$, which fullfills

$$\int \mathrm{d}x\, f(x)h(x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(\mathcal{F}[\phi+\varepsilon h] - \mathcal{F}[\phi])$$

for any function $h(x)$. The notation $\left.\frac{\delta\mathcal{F}[\phi]}{\delta\phi(x_1)}\right|_{\phi(x)}$ means the functional derivative $f(x)$ evaluated at x_1.

Taking the function derivative, this produces the correct correlator as expected

$$\begin{aligned}\langle \xi(t_1) \rangle &= \left[\Gamma J(t_1) Z[J] \right]_{J=0} = 0, \\ \langle \xi(t_1)\xi(t_2) \rangle &= \left[\Gamma \delta(t_1 - t_2) + \Gamma^2 J(t_1) J(t_2) Z[J] \right]_{J=0} = \Gamma \delta(t_1 - t_2).\end{aligned}$$

5.2 Martin-Siggia-Rose formalism

We are now ready to develop the path integral formulation of stochastic processes governed by Langevin equations. This allows us to carry over the intuition we built up in the path integral formulation of quantum mechanics and directly apply it to stochastic processes. The similarity between stochastic processes and quantum mechanics can already be found in our previous treatment. For example, recall the evolution equation from the first section as well as quantum mechanics' time evolution operator

$$\mathcal{P}(y_2, t + \Delta t) = \int \mathrm{d}y_1 \, \mathcal{P}(y_2, t + \Delta t | y_1, t) \mathcal{P}(y_1, t), \qquad \text{(Evolution equation)}$$

$$\langle y_2 | \psi(t + \Delta t) \rangle = \int \mathrm{d}y_1 \, \langle y_2 | U(t + \Delta t, t) | y_1 \rangle \langle y_1 | \psi(t) \rangle \qquad \text{(QM propagator)}$$

and note the similarity between the two equations. In this analogy the transition probabilities $\mathcal{P}(y_2, t + \Delta t | y_1, t)$ take the role of the propagator $U(t + \Delta t, t)$. In quantum mechanics, the propagator is given through a path integral, with each path weighted exponentially by a classical action, $U(t, t_0) = \int \mathcal{D}x \, \exp\left(\frac{i}{\hbar} \int_{t_0}^{t} \mathrm{d}t \, L(x, \dot{x}, t) \right)$. We will derive in the following a similar expression for the stochastic propagator $\mathcal{P}(y_2, t + \Delta t | y_1, t)$ and obtain a the so called Martin-Siggia-Rose action as the equivalent stochastic action.

Let's start by transforming the Langevin equation[4]

$$\dot{x} = F(x) + \zeta(t)$$

into a path integral, where for convenience we have set $\gamma = 1$. In order to find the path integral, first define $L[x, \dot{x}] = \dot{x} - F(x)$ and discretize the differential equation at the times $t_1, ..., t_n$

$$L_n := \frac{x_n - x_{n-1}}{\Delta t} - F(x_n) = \zeta_n$$

with $x_n := x(t_n)$ and $\zeta_n := \zeta(t_n)$. The choice of letter L is on purpose as this quantity will later turn out to take the role of a Lagrangian within the MSR formalism. We may then transform this discretized Langevin equation into a path integral by expanding 1 with

[4]Note that while in principle this looks like the overdamped Langevin equation, we may transform the general Langevin equation including an inertia term of second time derivative into a first order Langevin stochastic differential equation.

delta functions[5]

$$1 = \int_{\mathbb{R}^n} \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n \prod_{k=1}^{n} \Delta t \delta(L_k - \zeta_k) \to \int \mathcal{D}x \, \delta(L[x(t), \dot{x}(t)] - \zeta(t))$$

in the limit $n \to \infty$, where we take $t_1, ..., t_n$ from $-\infty$ to ∞. We use $\mathcal{D}$ to abbreviate the integration measure[6] $\mathcal{D}x = \prod_{i=1}^{N} \Delta t dx_i$. Physically, the above path integrals sums over all physical trajectories that satisfy the Langevin equation with given noise ζ. Because for given noise ζ Langevin's equation is deterministic, the path's probability weight is a delta function.

Next, we define a *partition function* Z for this setting by taking our previous path integral representations of solutions to the Langevin equation *given* a noise functional ζ and averaging it over all possible noise functionals

$$Z := \underbrace{\int \mathcal{D}\zeta \, \mathcal{P}[\zeta(t)]}_{\text{all possible noises}} \quad \cdot \quad \underbrace{\int \mathcal{D}x \, \delta(L[x, \dot{x}] - \zeta)}_{\text{Langevin trajectories for noise } \zeta} \quad .$$

We may now plug in our expression for $\mathcal{P}[\zeta(t)]$ from equation (18) and use the Fourier representation of the delta function $\delta(x) = \int \mathcal{D}p \, e^{\int \mathrm{d}t \, ipx}$

$$\begin{aligned} Z &= \int \mathcal{D}x \mathcal{D}p \mathcal{D}\zeta \, \exp\left(- \int dt \, \frac{1}{2\Gamma} \zeta^2 + 2ip(L[x, \dot{x}] + \zeta) \right) \\ &= \int \mathcal{D}x \mathcal{D}p \, \exp \underbrace{\left(-2i \int dt \, pL[x, \dot{x}] + i\Gamma p^2 \right)}_{=: -2iS[x,p]}, \end{aligned} \tag{19}$$

where last step is a Gaussian integration[7] over ζ. Physically, we extended the phase space to (x, p) where p has no actual physical meaning besides the intended resemblance to momentum as second classical phase variable. This reminds us of quantum mechanics: the partition function is a path integral weighted exponentially by a time integration, and motivates us by analogy to quantum mechanics to introduce an action for our setting, the *Martin-Siggia-Rose* (MSR) action

$$S[x, p] = \int \mathrm{d}t \, \left[L[x, \dot{x}] + i\Gamma p^2 \right] = \int \mathrm{d}t \, \left[p\dot{x} - pF(x, t) + i\Gamma p^2 \right]. \tag{20}$$

[5]Mathematically, one has to worry about the transformation $\delta(x_k - \zeta_k) \to \delta(L[x_k] - \zeta_k)$, i.e. take the determinant of the Jacobian $\det(\frac{\delta L}{\delta x})$ into account. However, by proper regularization (Ito) this determinant becomes 1. Also note that for a general (e.g. non-linear) stochastic ODE expand L in x, i.e. $L_i = L_{ij}x_j + \Gamma_{ijk}x_j x_k + ...$ with general Jacobian $J_{ij} = \frac{L_i}{dx_j} = L_{ij} + 2\Gamma_{ijk}x_k + ...$

[6]Throughout this document, we use the symbol $\mathcal{D}$ sloppily: different path integrals may include different normalization constant, even though we use the same sign for them. For example, here $\mathcal{D}x = \prod_{i=1}^{N} \Delta t \mathrm{d}x_i$, but in the following path integral $\mathcal{D}p = \prod_{i=1}^{N} \frac{dp_i}{\pi \Delta t}$

[7]Think about these continuous Gaussian integrations as the limit of a sum of Gaussian integrations. In this case we have the integration measure $\mathcal{D}\zeta = \prod_{i=1}^{n} \frac{d\zeta_i}{\sqrt{2\pi\Gamma\Delta t}}$, where Δt is the spacing of the time steps $t_1, ..., t_n$

The quantity Z is similar to a partition function from classical statistical physics in that it sums the complete state space of random noise trajectories ζ and corresponding Langevin trajectories x. Even though it always holds $Z = 1$, the partition function Z is an interesting quantity to study. For example, observables may be computed as

$$\langle \mathcal{O}[x(t)] \rangle = \int \mathcal{D}x \mathcal{D}p \; \mathcal{O}[x(t)] \; e^{-2iS[x,p]}.$$

We can integrate the auxiliary trajectory $p(t)$ out from our last representation of the partition function, equation (20)

$$Z = \int \mathcal{D}x \, e^{-\frac{1}{2\Gamma}\int \mathrm{d}t \, L^2[x,\dot{x}]} = \int \mathcal{D}x \, \exp\left(-\frac{1}{2\Gamma}\int dt \, (\dot{x} - F(x))^2\right).$$

This representation reveals that Z contains an additional stochastic interpretation similar to the one of quantum mechanics: since Z includes all noise terms and paths and integrates to 1, we may interpret a path integral over all paths starting at x_0 and ending in x as a transition probability

$$\mathcal{P}(x,t|x_0,t_0) = \int_{x_0}^{x} \mathcal{D}x \, \exp\left(-\frac{1}{2\Gamma}\int_{t_0}^{t} dt \, (L[x,\dot{x}])^2\right), \tag{21}$$

where this path integrals symbolically sums over all paths that begin in x_0 at t_0 and end in x at t. Notice the similarity to the general form of a quantum mechanical propagator

$$\langle x_f, t_f | x_0, t_0 \rangle = \int_{x_0}^{x_f} \mathcal{D}x \, \exp\left(\frac{i}{\hbar}\int_{t_0}^{t_f} \mathrm{d}t \, L[x,\dot{x}]\right), \tag{22}$$

which apart from the imaginary unit (a *Wick rotation*) in the exponential is qualitatively identical to our expression for the transition probability above.

5.3 Greens function with path integrals

Equipped with a path integral formulation of Langevin's equation, we now want to learn how this can be used to solve the Fokker-Planck equation. To do so, we derive Green's function for the diffusion in the overdamped limit (8) with the path integral formalism. Since the Green function is equal to the propagator we get

$$\begin{aligned} G(x,x';t,t') &= \int_{x'}^{x} \mathcal{D}x \, \exp\left(-\frac{1}{2\Gamma}\int_{t'}^{t} dt \, (\dot{x} - F(x))^2\right) \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi\Gamma\Delta t}} \int \prod_{i=1}^{n-1} \frac{dx_i}{\sqrt{2\pi\Gamma\Delta t}} \exp\left(\frac{\Delta t}{2\Gamma}\sum_{i=1}^{n}\left(\frac{(x_i - x_{i-1})}{\Delta t} - F(x_i)\right)^2\right). \end{aligned}$$

We compute these integrals for the case $F(x) = 0$

$$G(x, x'; t, t') = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi\sigma}} \int \prod_{i=1}^{n-1} \frac{dx_i}{\sqrt{2\pi\sigma}} \exp\left(\frac{1}{2\sigma^2} \sum_{i=2}^{n-1} (x_i - x_{i-1})^2 + (x_1 - x')^2 + (x - x_{n-1})^2 \right)$$

with $\sigma = \sqrt{\Gamma \Delta t}$. In order to evaluate the integral we need that

$$\begin{aligned} S(x_{k+1}, x_k, x') &:= (x_{k+1} - x_k)^2 + \frac{1}{k}(x_k - x')^2 \\ &= x_{k+1}^2 - 2x_{k+1}x_k + x_k^2(1 + \frac{1}{k}) - \frac{2}{k}x_k x' + \frac{1}{k}x'^2 \\ &= \frac{k+1}{k}\left(x_k - \left(\frac{k}{k+1}x_{k+1} + \frac{1}{k+1}x' \right) \right)^2 + \frac{1}{k+1}(x_{k+1} - x')^2. \end{aligned}$$

The single integration over x_k gives

$$\int dx_k \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2} S(x_{k+1}, x_k, x') \right) = \frac{\sqrt{k}}{\sqrt{k+1}} exp\left(-\frac{1}{2\sigma^2} \left(\frac{1}{k+1}(x_{k+1} - x')^2 \right) \right).$$

This integration can be used inductively to evaluate the integral above

$$\begin{aligned} G(x, x'; t, t') &= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi\Gamma\Delta t n}} \exp\left(-\frac{(x - x')^2}{2\Gamma n \Delta t} \right) \\ &= \frac{1}{\sqrt{2\pi\Gamma(t - t')}} \exp\left(-\frac{(x - x')^2}{2\Gamma(t - t')} \right). \end{aligned}$$

Here we see directly that the diffusion in the overdamped limit can be described by a Wiener process. Furthermore we can verify that the Brownian particle fullfills the Markov property, because the Wiener process is a Markov process.

5.4 Path integral derivation of the Fokker-Planck equation

The path integral representation of the Langevin equation also allows for an alternate derivation of the Fokker-Planck equation, analogously to the derivation of the Schroedinger equation from from quantum mechanical path integrals.

Let's evolve our probability density function $\mathcal{P}(x, t)$ for a small time step Δt

$$\begin{aligned} \mathcal{P}(x, t + \Delta t) &= \int \mathrm{d}y \, \mathcal{P}(x, t + \Delta t | y, t) \mathcal{P}(y, t) \\ &= \int \mathrm{d}(\delta x) \, \mathcal{P}(x, t + \Delta t | x - \delta x, t) \mathcal{P}(x - \delta x, t). \end{aligned}$$

Notion	Quantum Mechanics	Stochastics (MSR)
State space	location x and momentum p	location x and auxiliary variable p
Central Object	State $\Psi(x,t)$	Probability $\mathcal{P}(x,t)$
Propagator	$\langle x_f, t_f \vert x_0, t_0\rangle$	$\mathcal{P}(x,t\vert x_0,t_0)$
Hamiltonian	$H(x,p)$	$H[x,p] = pF(x) - i\Gamma p^2$
Action	$\int \mathrm{d}t\, L(x,\dot{x})$	$\int \mathrm{d}t\, L^2[x,\dot{x}]$
Time evolution	Schroedinger eq	Fokker-Planck eq

Table 2: Analogy between various statistical notions introduced in this document and the corresponding quantum mechanical notions.

From equation (21) we know the propagator $\mathcal{P}(x,t+\Delta t|x-\delta x,t)$

$$\mathcal{P}(x,t+\Delta t) = \int \frac{d(\delta x)}{\sqrt{2\pi\Delta t\Gamma}} \exp\left(-\frac{1}{2\Gamma}\Delta t\left(\frac{\delta x}{\Delta t} - F(x-\delta x)\right)^2\right)\mathcal{P}(x-\delta x,t).$$

By expanding the square in the exponential above and further Taylor expanding the terms as well as the probability distribution[8], we obtain

$$\begin{aligned}\mathcal{P}(x,t+\Delta t) &= \int \frac{d(\delta x)}{\sqrt{2\pi\Gamma\Delta t}} \exp\left(-\frac{(\delta x)^2}{2\Gamma\Delta t}\right)\left(1+\delta x\frac{1}{\Gamma}F(x) - \frac{1}{\Gamma}(\delta x)^2\frac{\partial F(x)}{\partial x} + \mathcal{O}\left((\delta x)^3\right)\right)\\ &\quad\times(1+\mathcal{O}(\Delta t))\left(\mathcal{P}(x,t) - \delta x\frac{\partial\mathcal{P}(x,t)}{\partial x} + \frac{1}{2}(\delta x)^2\frac{\partial^2\mathcal{P}(x,t)}{\partial x^2} + \mathcal{O}\left((\delta x)^3\right)\right)\\ &= \mathcal{P}(x,t) + \left(\frac{\Gamma}{2}\frac{\partial^2\mathcal{P}(x,t)}{\partial x^2} - F(x)\frac{\partial\mathcal{P}(x,t)}{\partial x} + \frac{\partial F(x)}{\partial x}\mathcal{P}(x,t)\right)\Delta t + \mathcal{O}((\Delta t)^2)\end{aligned}$$

By taking the limit $\Delta t \to 0$ we get the Fokker-Planck equation

$$\frac{\partial\mathcal{P}(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[F(x)\mathcal{P}(x,t) - \frac{\Gamma}{2}\frac{\partial\mathcal{P}(x,t)}{\partial x}\right].$$

This derivation was analogous to the derivation of the Schroedinger equation $i\hbar\partial_t\Psi(x,t) = \hat{H}\Psi(x,t)$ and the Fokker-Planck equation might even be recasted into a similar form

$$\frac{i}{2}\partial_t\mathcal{P}(x,t) = H[x,p]\mathcal{P}(x,t)$$

with Hamiltonian $H(x,p) = pF(x) - i\Gamma p^2$ by identifying the MSR auxiliary variable $p \to -\frac{i}{2}\partial_x$ using first a quantization approach from quantum mechanics.

The analogy between quantum mechanics and stochastic systems in the MSR formalism now encompasses a state space, Hamiltonian, Lagrangian, action, and a propagator, see also

[8]We expan the mixed term $\propto e^{\delta x F(x-\delta x)}$ and the probability function $\mathcal{P}(x-\delta x,t)$ around x as well as the quadratic in the force term in Δt, i.e. $\propto e^{\Delta t F^2(x-\delta x)} \approx 1+\mathcal{O}(\Delta t)$. The Gaussian integration in odd powers of δx vanish, the Gaussian integration with δx^2 already results in a quadratic term in Δt.

table 2. The MSR formalism works such that most tools from physics can simply be carried over and used analogously. For example, one might try to analyze the MSR Hamiltonian $H[x, p]$ for minimal energy trajectories $(x(t), p(t))$. This completes our analogy of quantum mechanics and stochastic processes in a path integral formulation. Besides the beauty of this analogy, the language itself will help us to understand the field theoretic derivation of Langevin's equation in the chapter.

6 Keldysh formalism

This chapter aims to give a conceptual introduction to the *Keldysh formalism*, a non-equilibrium quantum field theoretic framework, following the presentation from Kamenev [7]. Using the Keldysh formalism we will be able to recover the Langevin and Fokker-Planck equations as a classical limit from quantum field theory. Even though the Keldysh formalism itself is an advanced theory, we have built up all many tools necessary to understand the physics behind it.

6.1 Closed time contour and partition function

Before we delve into the Keldysh formalism, we need to summarize some concepts from quantum statistical mechanics. The time evolution of a many-body mixed state ρ governed by the time-dependent Hamiltonian $H(t)$ is described by the von Neumann equation

$$\partial_t \rho(t) = -i\hbar \left[H(t), \rho(t)\right].$$

A solution to the von Neumann equation uses the time ordered *evolution operator* $\mathcal{U}_{t,-\infty}$ that shifts the initial state $\rho(-\infty)$ at time $-\infty$ to time t

$$\rho(t) = \mathcal{U}_{t,-\infty}\rho(-\infty)\mathcal{U}_{-\infty,t},$$

where we assume that at time $t = -\infty$ the particles are non-interacting. The explicit form of the time evolution operator $\mathcal{U}_{t,t'}$ can be found to be

$$\mathcal{U}_{t,t'} = \mathbb{T}\exp\left(-\frac{i}{h}\int_{t'}^{t} \mathrm{d}t\, H(t)\right),$$

where $\mathbb{T}$ denotes the time-ordering operator. The expectation value of an observable $\mathcal{O}$ can be expressed using the evolution operator

$$\langle\mathcal{O}\rangle(t) = \frac{\mathrm{Tr}\left\{\mathcal{O}\rho(t)\right\}}{\mathrm{Tr}\left\{\rho(t)\right\}} = \frac{\mathrm{Tr}\left\{\mathcal{U}_{-\infty,t}\mathcal{O}U_{t,-\infty}\rho(-\infty)\right\}}{\mathrm{Tr}\left\{\rho(-\infty)\right\}},$$

where we used the cyclic permutation property of the trace.

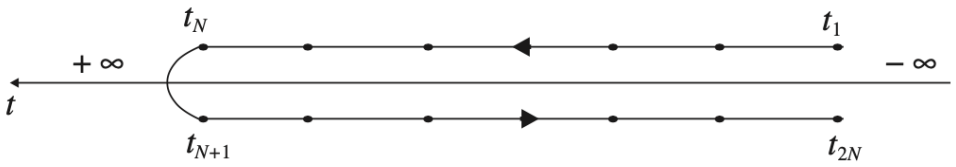


Figure 5: The Keldysh contour $\mathcal{C}$. The evolution operators promote a state from time $-\infty$ (right) to $+\infty$ and back to $-\infty$. To derive a path integral formulation of the Keldysh partition function, we discretize this contour.

Similarly to our discussion in the path integral chapter we can define a partition function Z in analogy to the observable average as a sum of the entire sample space, which here is a time shift from $-\infty$ to ∞ and back to $-\infty$, in equations

$$Z := \frac{\mathrm{Tr}\left\{\mathcal{U}_{-\infty,\infty}\mathcal{U}_{\infty,-\infty}\rho(-\infty)\right\}}{\mathrm{Tr}\left\{\rho(-\infty)\right\}} = \frac{\mathrm{Tr}\left\{\mathcal{U}_{\infty,-\infty}\rho(-\infty)\mathcal{U}_{-\infty,\infty}\right\}}{\mathrm{Tr}\left\{\rho(-\infty)\right\}} = 1.$$

The second expression we can again interpret as an evolution from the system from $-\infty$ to ∞. With this combination of time evolution operators $\mathcal{U}_{-\infty,\infty}\mathcal{U}_{\infty,-\infty}$ we encounter the first central object of the Keldysh formalism, the so called *closed time contour* $\mathcal{C}$. This contour is visualized in figure 5.

In statistical mechanics we like partition functions, because we can express quantities of interested as derivatives of the partition functions. Let us briefly motivate how we can do the same for our partition function Z from above. Let $\mathcal{O}$ be an observable and $V(t)$ be an arbitrary function. Define a Hamiltonian for the forward and backward contour $H_V(t)^{\pm} = H_0(t) \pm \mathcal{O}V(t)$ and the generating function analogously to the partition function

$$Z[V] := \frac{\mathrm{Tr}\left\{\mathcal{U}^{V^-}_{-\infty,\infty}\mathcal{U}^{V^+}_{\infty,-\infty}\rho(-\infty)\right\}}{\mathrm{Tr}\left\{\rho(-\infty)\right\}}.$$

The expectation value of the observable $\mathcal{O}$ can now be obtained by a functional derivative

$$\langle\mathcal{O}\rangle(t_1) = \frac{i\hbar}{2}\frac{\delta Z[V]}{\delta V(t_1)}\bigg|_{V=0}.$$

6.2 Keldysh path integral formulation of an harmonic oscillator

As a next step, we are interested to derive a path integral formulation of our partition function Z. For convenience we consider a second quantized formulation of the harmonic oscillator with annihilation and creation operators b, $b^\dagger$ that act on the number states $|n\rangle$. In this formalism the Hamiltonian is given by

$$H = \omega_0 b^\dagger b.$$

We are interested in the coherent states of the harmonic oscillator

$$|\varphi\rangle = \sum_{n=0}^{\infty} \frac{\varphi^n}{\sqrt{n!}} |n\rangle,$$

which are eigenstates of the annihilation operator b with eigenvalue φ. The eigenvalue φ can be an arbitrary complex number. The coherent states $|\varphi\rangle$ form an overcomplete basis and the unity operator [9] is expressed as

$$1 = \int \mathrm{d}[\varphi, \overline{\varphi}]\, e^{-|\varphi|^2} |\varphi\rangle\langle\varphi|. \tag{23}$$

To obtain the path integral formulation we discretize both, the forward and backward parts of the Keldysh time contour, in $-\infty = t_1, t_2, ..., t_n = \infty$ and $\infty = t_{n+1}, t_{n+2}, ..., t_{2n} = -\infty$, respectively, see also figure 5, as well as the the evolution operators

$$\mathcal{U}_{\infty,-\infty} = \prod_{k=1}^{n-1} \mathcal{U}_{t_{k+1},t_k} \quad \text{and} \quad \mathcal{U}_{-\infty,\infty} = \prod_{k=n}^{2n-1} \mathcal{U}_{t_{k+1},t_k}.$$

By inserting the resolution of unity, equation (23), for each time step we can expand the trace

$$\mathrm{Tr}\left\{\mathcal{U}_{-\infty,\infty}\mathcal{U}_{\infty,-\infty}\rho(-\infty)\right\} = \int \mathrm{d}\varphi_1 ... \mathrm{d}\,\varphi_{2n} \prod_{k=1}^{2n-1} e^{-|\varphi_k|^2} \langle\varphi_{k+1}|\mathcal{U}_{t_{k+1},t_k}|\varphi_k\rangle\langle\varphi_1|\rho(-\infty)|\varphi_{2n}\rangle.$$

To derive a path integral expression we consider small time steps $\delta t = t_{k+1} - t_k \to 0$. After some further mathematical manipulations we arrive at the path integral formulation of the partition function Z

$$Z = \frac{1}{\mathrm{Tr}\left\{\rho(-\infty)\right\}} \int \mathcal{D}[\varphi, \overline{\varphi}]\, e^{\frac{i}{h} S[\varphi,\overline{\varphi}]}$$

with action

$$S[\varphi, \overline{\varphi}] = \int_{\mathcal{C}} \mathrm{d}t\, \overline{\varphi}(i\partial_t - \omega_0)\varphi. \tag{24}$$

Note that the time integration is along the time contour $\mathcal{C}$. To simplify this contour into the real line we divide the fields φ into a their parts on the forward and backward branch, $\varphi^{\pm}$, so that we can get

$$S[\varphi, \overline{\varphi}] = \int_{-\infty}^{\infty} \mathrm{d}t \;\left[\overline{\varphi}^{+}(i\partial_t - \omega_0)\varphi^{+} - \overline{\varphi}^{-}(i\partial_t - \omega_0)\varphi^{-}\right].$$

Note that the here used continuous notation is simply an abbreviation arising from the path integral formulation.

[9] We use the complex integration measure $\mathrm{d}[\varphi, \overline{\varphi}] = \frac{\mathrm{dRe}(\varphi)\mathrm{dIm}(\varphi)}{\pi}$

For the following discussion assume the initial state $\rho(-\infty)$ to be in thermal equilibrium

$$\rho(-\infty) = e^{-\beta(H-\mu N)},$$

where β is the inverse temperature and μ the chemical potential. Generally, this condition does not restrict us from treating non-equilibrium systems, because the perturbations can be switched on and off during the evolution on the time contour, but with the particular Hamiltonian of the harmonic oscillator considered in this section the following results only hold in for an initial thermal equilibrium.

The second central ingredient of the Keldysh formalism is the *Keldysh rotation*, defined by the coordinate transformation

$$\varphi^{cl} = \frac{1}{\sqrt{2}}(\varphi^+(t) + \varphi^-(t)) \quad \text{and} \quad \varphi^q = \frac{1}{\sqrt{2}}(\varphi^+(t) - \varphi^-(t)).$$

The superscript cl and q stand for the classical and quantum components of the fields φ. Without proof, the Keldysh rotated action for the harmonic oscillator takes the following Gaussian form

$$S_K[\boldsymbol{\varphi}, \overline{\boldsymbol{\varphi}}] = \iint \mathrm{d}t\mathrm{d}t' \, \boldsymbol{\varphi}^*(t) G^{-1}(t,t') \boldsymbol{\varphi}(t'),$$

where the classical and quantum component form the vector $\boldsymbol{\varphi}(t) = (\varphi^{cl}(t), \varphi^q(t))$. The matrix G of the quadratic form in the action is given by

$$G(t,t') = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix} = -ie^{-i\omega_0(t-t')} \begin{pmatrix} 2n_B(\omega_0)+1 & \theta(t-t') \\ -\theta(t'-t) & 0 \end{pmatrix},$$

with the bosonic occupation number $n_B = \frac{e^{-\beta(\omega_0-\mu)}}{1-e^{-\beta(\omega_0-\mu)}}$, which is a consequence of the initial thermal equilibrium of $\rho(-\infty)$. By applying a Fourier transform on G with respect to $t-t'$ we get the energy representation of the matrix G

$$G^K(\varepsilon) = -2\pi i[2n_n(\varepsilon)+1]\delta(\omega_0-\varepsilon) \quad \text{and} \quad G^{R(A)} = (\varepsilon - \omega_0 \pm i0)^{-1}.$$

The matrix G is called *Green's function*, because up to a constant the elements of the matrix are equal to the correlators

$$\langle \varphi(t)^\alpha \, \overline{\varphi}^\beta(t') \rangle = \int \mathcal{D}[\boldsymbol{\varphi}, \overline{\boldsymbol{\varphi}}] \, \varphi(t)^\alpha \overline{\varphi}^\beta(t') e^{-\frac{i}{\hbar}S_K} = i\hbar G^{\alpha,\beta}(t,t')$$

for $\alpha, \beta = (cl, q)$. These correlators are called Green's function, because they describe the response of the system to the creation and annihilation of a quantum.

6.3 Single particle Keldysh action

In this section we want to use the formalism of the previous section for real fields. In analogy to the treatment of the harmonic oscillator in quantum mechanics, we split $\phi(t)$

into real and imaginary part

$$\phi(t) = \frac{1}{\sqrt{2\omega_0}}(P(t) - i\omega_0 X(t))$$

With this expression for $\phi(t)$ we can rewrite the action from equation (24) in terms of $X(t)$ and $P(t)$

$$S[X,P] = \int_{\mathcal{C}} \mathrm{d}t \, \left[P\dot{X} - \frac{1}{2}\dot{X}^2 + \frac{\omega^2}{2}X^2 \right].$$

This is the action of a classical harmonic oscillator with position $X(t)$ and momentum $P(t)$. Integrating out momenta in the path integral formulation gives

$$S[X] = \int_{\mathcal{C}} \mathrm{d}t \, \left[\frac{1}{2}\dot{X}^2 - \frac{\omega^2}{2}X^2 \right].$$

This can be generalised for a particle in arbitrary potential $V(X)$ by replacing $\frac{\omega^2}{2}X^2$ with $V(x)$. As before we perform a Keldysh rotation

$$X^{cl}(t) = \frac{1}{2}[X^+(t) + X^-(t)] \quad \text{and} \quad X^q(t) = \frac{1}{2}[X^+(t) - X^-(t)].$$

Finally, we arrive at the *quantum single particle Keldysh action*

$$S[X^{cl}, X^q] = -\int \mathrm{d}t \, 2X^q(\ddot{X}^{cl} + V'(X^{cl})), \tag{25}$$

where we only consider terms up to first order in X^q. The Keldysh action of the harmonic oscillator in thermal equilibrium can be recast into the familiar form

$$S[X^{cl}, X^q] = \int_{-\infty}^{\infty} \mathrm{d}t \, X^T D^{-1} X \tag{26}$$

with Green's function matrix D. In energy representation, the matrix D is given by

$$\begin{aligned} D^{R(A)}(\varepsilon) &= \frac{1}{2}\frac{1}{(\partial_t \pm i0)^2 - \omega_0^2}, \\ D^K(\varepsilon) &= \coth\left(\frac{\varepsilon}{2T}\right)(D^R(\varepsilon) - D^A(\varepsilon)), \end{aligned} \tag{27}$$

where T is the temperature of the system.

6.4 Many particle Brownian Keldysh action

Now we are prepared to derive the Langevin equation. For this, we promote our single particle discussion of the previous sections to the Brownian setting, i.e. a large Brownian particle in a bath of many small particles. The environment of the Brownian particle is

modelled as a bath of harmonic oscillators in thermal equilibrium with degree of freedom φ_s. We divide the action into three parts, the Brownian particle action $S[X]$ from before (25), the action of the bath of smaller particles $S_{\text{bath}}[X, \varphi_s]$, a sum over many single particle actions (26), and the interactions between smaller and Brownian particles $S_{\text{int}}[X, \varphi_s]$, in equations

$$\begin{aligned} S[X] &= -\int \mathrm{d}t\, 2X^q(\ddot{X}^{cl} + V'(X^{cl})), \\ S_{\text{bath}}[X, \varphi_s] &= \frac{1}{2} \sum_{s \text{ in bath}} \int_{-\infty}^{\infty} \mathrm{d}t\, \varphi_s^T D_s^{-1} \varphi_s, \\ S_{\text{int}}[X, \varphi_\omega] &= \sum_{s \text{ in bath}} g_s \int_{-\infty}^{\infty} \mathrm{d}t\, X^T \sigma_1 \varphi_s, \end{aligned}$$

where σ_1 denotes the first Pauli matrix. We call the dissipative action the sum of the bath action and interaction action. After integrating out the degrees of freedom of the bath φ_s this dissipative action is given by

$$\begin{aligned} S_{\text{disspative}}[X] &= \frac{1}{2} \int \int \mathrm{d}t \mathrm{d}t'\, X^T D^{-1} X, \\ D^{-1}(t - t') &= -\sigma_1 \sum_{s \text{ in bath}} g_s^2 D_s^{-1}(t - t') \sigma_1. \end{aligned} \tag{28}$$

From equation (27) we can determine the energy representation of D^{-1}

$$[D_s^{-1}]^{R(A)} = -\frac{1}{2} \sum_{s \text{ in bath}} \frac{g_s^2}{(\varepsilon \pm i0)^2 - \omega_s^2} = \int_0^{\omega} \frac{\mathrm{d}\omega}{2\pi} \frac{\omega J(\omega)}{\omega^2 - (\varepsilon \pm i0)^2},$$

where $J(\omega)$ is the spectral density defined as

$$J(\omega) = \pi \sum_{s \text{ in bath}} \frac{g_s^2}{\omega_s} \delta(\omega_s - \omega).$$

The spectral density sums all frequencies ω that are present in the bath of harmonic oscillators. In the following we will assume the simplifying *ohmic bath model*, in which the spectral density is given by $J(\omega) = 4\gamma\omega$. Now we can derive an explicit expression of D

$$\begin{aligned} [D^{-1}]^{R(A)} &= C \pm 2i\gamma\varepsilon, \\ [D^{-1}]^{K} &= 4i\gamma\varepsilon \coth\left(\frac{\varepsilon}{2T}\right), \end{aligned}$$

where C is a constant. If we transform D back into the time representation and plug into the dissipative action (28), we find the final *quantum Brownian Keldysh action*

$$\begin{aligned} S[X] = \int \mathrm{d}t\, &\left[-2X^q(\ddot{X}^{cl} + \gamma\dot{X}^{cl}) - V(X^{cl} + X^q) + V(X^{cl} - X^q)\right] \\ &+2i\gamma \int \mathrm{d}t\, \left[T(X^q)^2 + \frac{\pi T^2}{2} \int \mathrm{d}t'\, \frac{(X^q(t) - X^q(t'))^2}{\sinh^2[\pi T(t - t')]}\right]. \end{aligned} \tag{29}$$

6.5 Classical limit of Brownian Keldysh action and Langevin equation

By rescaling the component $X^q \to \hbar X^q, T \to T/\hbar$ in the Keldysh action (29) and taking the classical limit $\hbar \to 0$, we find the *classical Brownian Keldysh action*

$$S[X] = \int \mathrm{d}t \, \left[-2X^q[\ddot{X}^{cl} + \gamma \dot{X}^{cl} + V'(X^{cl})] + 4i\gamma T (X^q)^2 \right]. \tag{30}$$

This action has partition function $Z = \int \mathcal{D}X^q \mathcal{D}X^{cl} \, e^{iS[X]}$. In the MSR formalism, we postulated the Langevin equation and derived from it the MSR action (20). In contrast, the above action (30) was derived from quantum mechanical considerations, so we can now perform the inverse steps of our MSR derivation, equation (19), to find out if our classical Brownian Keldysh action reveals an equation in turn.

First, we use the *Hubbard-Stratonovich transformation*

$$\exp\left(-(X^q)^2/2a\right) = \sqrt{\frac{1}{2\pi a}} \int \mathrm{d}\zeta \, \exp\left\{ -\frac{\zeta^2}{2a} - iX^q\zeta \right\},$$

to introduce an auxiliary field ζ

$$\begin{aligned} Z &= \int \mathcal{D}X^{cl} \mathcal{D}X^q \, e^{iS[X]} \\ &= \int \mathcal{D}\zeta \mathcal{D}X^{cl} \, e^{-\frac{1}{4\gamma T}\int \mathrm{d}t\, \zeta^2} \int \mathcal{D}X^q \, \exp\left\{ -2i \int \mathrm{d}t \, X^q(\ddot{X}^{cl} + \gamma \dot{X}^{cl} + V'(X^{cl}) - \zeta) \right\}. \end{aligned}$$

Observe that the known probability for a ζ trajectory $\mathcal{P}[\zeta(t)] = \exp\left(-\int \mathrm{d}t \, \frac{\zeta^2(t)}{2\Gamma} \right)$, equation (18), already pops up here. If we reduce the right part in the above partition function to a delta function

$$Z = \int \mathcal{D}\zeta \mathcal{D}X^{cl} \, e^{-\frac{1}{4\gamma T}\int \mathrm{d}t\, \zeta^2} \, \delta\left(\ddot{X}^{cl} + \gamma \dot{X}^{cl} + V'(X^{cl}) - \zeta) \right)$$

we find that in this partition function only trajectories $X^{cl}(t)$ satisfying the equation in the delta function above

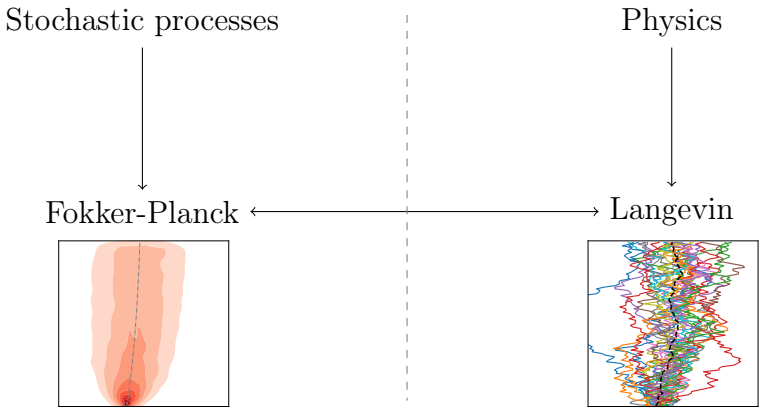
$$\ddot{X}^{cl} = -\gamma \dot{X}^{cl} - V'(X^{cl}) + \zeta(t)$$

enter. This is the Langevin equation we postulated in section 3, but derived here from quantum field theoretic considerations. Here, the classical Keldysh component X^{cl} takes the role of location x (hence the name classical), the function ζ the role of a random force and the negative derivative of the potential $-V'$ the role of the external forces F. The correlator of the random force is

$$\langle \zeta(t)\zeta(t') \rangle = \int \mathcal{D}\zeta \, \zeta(t)\zeta(t') e^{-\frac{1}{4\gamma T}} = 2\gamma T \delta(t - t')$$

and hence reproduces the correlator (4) we postulated in section 3, including the relation to temperature we found earlier using the Einstein stokes relation (10). This concludes our derivation of the Langevin equation in the Keldysh formalism.

7 Concluding remarks



Let's take a moment and summarize the take home messages. First, we learnt about the notion of a stochastic process and one special case of stochastic processes, the Markov process that has no memory. Within this purely mathematical theory we derived the Fokker-Planck equation for a Markov process. The Fokker-Planck equation describes the time evolution of the probability density of its Markov process. To learn about the physics of the Fokker-Planck equation, we postulated the Langevin equation as an stochastic extension of Newton's equation. This allowed us connect the worlds of classical mechanics and stochastic processes by computing the stochastic $D^{(n)}$ factors, the linearizations of the moments of the transition probability M_n. Next, we took this connection one step further and demonstrated how analogous the worlds of stochastic processes and quantum mechanics can be formulated in the path integral approach of Martin, Siggia and Rose. Finally, we were able to use the intuitions from this path integral language to derive the Langevin equation as a classical limit from non-equilibrium quantum field theory in the Keldysh formalism.

A Appendix

A.1 Stochastic differential equations

The Langevin equation is an example of a stochastic differential equation. Since $\zeta(t)$ is random, also the solutions $x(t)$ of the Langevin equation will be random variables. Consider a general stochastic differential equation

$$\dot{X}_t = f(X_t, t) + \zeta_t, \qquad \zeta_t \sim \mathcal{N}(0, \sigma^2).$$

Note that again $\langle \zeta(t)\zeta(t') \rangle = \sigma^2 \delta(t - t')$. For an infinitesimal time step Δt, the equation reads

$$\mathrm{d}X_t = f(X_t, t)\mathrm{d}t + \mathrm{d}W_t, \qquad \mathrm{d}W_t = \int_t^{t+\Delta t} \mathrm{d}t' \, \zeta_{t'}.$$

Here, we encounter a stochastic integral for $\mathrm{d}W_t$. Intuitively, think of it as a sum of random variables. A sum of Gaussian random variables is again Gaussian, $\mathrm{d}W_t$ still has vanishing mean, $\langle \mathrm{d}W_t \rangle = 0$. Using $\langle \zeta_t \zeta_{t'} \rangle = \sigma^2 \delta(t - t')$ we find the second moment

$$\langle (\mathrm{d}W_t)^2 \rangle = \int_t^{t+\mathrm{d}t} \mathrm{d}t_1 \int_t^{t+\mathrm{d}t} \mathrm{d}t_2 \, \langle \zeta_{t_1}) \zeta_{t_2} \rangle = \sigma^2 \mathrm{d}t$$

Note that the second moment is linear in $\mathrm{d}t$, i.e. $\mathrm{d}W$ is of order $\sqrt{\mathrm{d}t}$. One thus often writes $\mathrm{d}W = \sigma\sqrt{\mathrm{d}t}$. This is an interesting property of stochastic integrals.

A.2 Langevin simulation code

Langevin trajectories are easy to simulate. Here we post the Python code used to generate the Langevin equation of section 3.3, velocity diffusion, which are plotted in figure 1. Please note the $\sqrt{\mathrm{d}t}$ dependence described in the above appendix also in the code.

```
import numpy as np

gamma, Gamma, m = 0.15, 1.2, 1

# time mesh to discretize [0, endtime]
# with n_timesteps many steps
n_timesteps = 100
endtime     = 10
dt          = endtime / n_timesteps

# define discretized trajectory including initial condition
trajectory    = np.zeros(n_timesteps)
```

```
trajectory[0] = -1

# integrate langevin trajectory
for timestep in range(1, n_timesteps):
    noise = np.random.normal(0, np.sqrt(Gamma))
    trajectory[timestep]  = trajectory[timestep-1]
    trajectory[timestep] -= dt * gamma / m * trajectory[timestep-1]
    trajectory[timestep] += np.sqrt(dt) * noise
```

A.3 Why Fokker-Planck equations cannot be solved by separation

We make a separation ansatz $\mathcal{P}(x,t) = X(x)T(t)$. The diffusion equation (5) transforms to

$$\frac{\dot{T}}{T} = \frac{\Gamma}{2\gamma^2}\frac{\ddot{X}}{X} =: \omega$$

These differential equations have the general solutions

$$\mathcal{P}(x,t) = X(x)T(t) = e^{\omega t}\left(ae^{-kx} + be^{kx}\right),$$

where $k = \sqrt{2\gamma^2\omega/\Gamma}$. For $\omega \neq 0$ these solutions are unbounded. This means they can not be interpreted as a probability density.

This is a general problem of the Separationsansatz. From the normalization condition $1 = \int dx X(x)T(t)$ we get that $T(t) = \left(\int dx X(x)\right)^{-1}$. Thereby the Separationansatz only leads to stationary solutions. This correspond to setting $\omega = 0$ in the example above.

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