Nonlinear Dynamics and Chaos I

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1 Introduction

Dynamical system is the tripel (P, E, F), where

1. P is the phase space for the dynamic variable $x \in P$,
2. E is the space of the evolutionary variable $t \in E$,
3. $F : P \times E \to P$ is the evolution rule.

We call (P, E, F) a **discrete dynamical system** (DDS) if $E = \mathbb{Z}$, and likewise a **continuous dynamical system** (CDS) if $E = \mathbb{R}$.

Types of evolutions

1. (**DDS**) Iterated mappings $x_{n+1} = F(x_n, n)$. For $\frac{\partial F}{\partial n} = 0$ it also holds that $x_{n+1} = F^n(x_0)$.
2. (**CDS**) First-order (system) of ODE $\dot{x} = f(x, t)$ with initial condition $x(t_0) = x_0$. For **autonomous systems** $f = f(x)$ the solution depends only on elapsed time and we define $x(x; t_0, x_0) = x(s - t_0, 0, t_0) =: x(t, x_0)$.

Flow map for an IVP with solution $t \mapsto \varphi(t; t_0, x_0)$ is the map

$$F_{t_0}^t(x_0) = \varphi(t; t, x_0).$$

For autonomous systems, we drop the initial time and write $F^t := F_0^t$.

Flow map properties

1. (**Smoothness**) $F_{t_0}^t$ is as smooth as $f(x, t)$.
2. (**Group property**) $F_{t_0}^{t_0} = Id, F_{t_0}^{t_2} = F_{t_1}^{t_2} \circ F_{t_0}^{t_1}$.
3. (**Inverse**) $(F_{t_0}^t)^{-1} = F_t^{t_0}$

2 Fundamentals

2.1 Peano's theorem

Peano's theorem If $f \in C^0$ near (x_0, t_0), then there exists a local solution $\varphi(t)$, i.e. $\dot{\varphi} = f(\varphi(t), t)$, $\varphi(t_0) = x_0$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

2.2 Picard's theorem

Picard's theorem Assume that $f \in C^0$ in t near (t_0, x_0) and that f is locally Lipschitz in x near (x_0, t_0). Then there exists a unique local solution to the IVP.

2.3 Geometric consequences of uniqueness

Trajectories in autonomous systems cannot intersect. Trajectories meet, but do not intersect in fixed points.

Trajectories in non-autonomous systems can intersect. Extend the phase plane by $\dot{t} = 1$, then there will be no intersection.

2.4 Local vs global uniqueness

Global existence If a local solution cannot be continued up to time T, then we must have $|x(t)| \to \infty$ as $t \to T$. Conversely, if a local solution is bounded, it can be extended globally.

Global existence in linear systems For a linear system $\dot{x} = A(t)x$, $A \in \mathbb{R}^{n \times n}$, $A \in C_t^0$. If $t \mapsto A(t)$ is bounded, i.e. its largest eigenvalue bounded, then a solution x exists globally,

$$|x(t)| \leq |x(t_0)| \exp\left(\int_{t_0}^t \lambda_{max}(s)\, \mathrm{d}s\right).$$

2.5 Dependence on initial conditions

Inherited regularity for initial conditions Consider the IVP $\dot{x} = f(x, t)$ with $x(t_0) = x_0$. If $f \in C_x^r$ for $r \geq 1$, then $x(t; t_0, x_0)$ is C^r in x_0 and the flow map $F_{t_0}^t$ is a C^r-diffeomorphism, i.e. $(F_{t_0}^t)^{-1}$ is also C^r.

Geometric meaning: images of the flow map are smoothly deformed.

Cauchy-Green strain tensor

$$C_{t_0}^t(x_0) = (\mathrm{D}F_{t_0}^t(x_0))^T\, \mathrm{D}F_{t_0}^t(x_0)$$

FTLE: finite time Lyaponov exponent

$$\mathrm{FTLE}_{t_0}^t(x_0) = \frac{1}{2(t - t_0)} \log \lambda_n(x_0),$$

where $\lambda_n(x_0)$ is the largest eigenvalue of the Cauchy-Green strain tensor. This largest eigenvalue typically grows exponentially.

Derivation Taylor $\xi(t) = x(t; t_0, x_0) - x(t; t_0, \tilde{x}_0)$, take squared norm $|\xi(t)|^2$ and take maximum $\max_{x_0, \xi_0} \frac{|\xi(t)|^t}{|\xi_0|^2}$

2.6 Dependence on parameters

Inherited regularity for parameters For an IVP $\dot{x} = f(x, t, \mu)$ with $x(t_0) = x_0$ and parameter μ. Define $X = (x, \mu)^T \in \mathbb{R}^{n+p}$, $F(X) = (f, 0)^T$, and $X_0 = (x_0, \mu_0)$. Then the previous results apply to the system $\dot{X} = F(X)$ with $X(t_0) = X_0$, i.e. $f \in C^r_{x,\mu}$ for $r \geq 1$ then $X(t)$ is C^r in X_0 and $x(t; t_0, x_0, \cdot) \in C^r_\mu$.

Lindstedt's approximation for periodic oscillations of nonlinear systems. Seek solutions of ansatz

$$\begin{cases} x_\epsilon(t) = \varphi_0(t;\epsilon) + \epsilon\varphi_1(t;\epsilon) + \mathcal{O}(\epsilon^2) \\ \varphi_i(t) = \varphi_i(t + T_\epsilon) \end{cases}$$

Define

1. *rescaled period*: $T_\epsilon = \frac{2\pi}{\omega(\epsilon)}$, $\omega(\epsilon) = 1 + \epsilon\omega_1 + \mathcal{O}(\epsilon^2)$,
2. *rescaled time* $\tau = \omega(\epsilon)t$, thus $\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{1}{\omega(\epsilon)}\frac{\mathrm{d}}{\mathrm{d}t}$
3. *rescaled ODE* $\frac{\mathrm{d}}{\mathrm{d}t} \to \omega(\epsilon)\frac{\mathrm{d}}{\mathrm{d}\tau}$.

Plug Ansatz into rescaled ODE and match equal powers of ϵ, i.e. groups (1), $\mathcal{O}(\epsilon)$, etc terms together. Choose period ω_1 such that resonance cancels.

Remark on epsilon dependence Note that the epsilon dependence in the period is essential. Solutions to Ansatzes of the form $x_\epsilon(t) = \varphi_0(t) + \epsilon\varphi_1(t) + \ldots + \mathcal{O}(\epsilon^r)$ might not be periodic due to resonance.

3 Stability of fixed points

3.1 Basic definitions

0-fixed points for $\dot{x} = f(x, t)$, $x \in \mathbb{R}^k, f \in C^r$, assume that $x = 0$ is a fixed point. Otherwise shift system by fix point to make it 0.

Lyapanov stability, stable point A point $x = 0$ is called stable if for all t_0 and $\epsilon > 0$ there exists $\delta = \delta(t_0, \epsilon) > 0$ such that for all $x_0 \in \mathbb{R}^k$ with $|x_0| \leq \delta$ we have $|x(t; t_0, x_0)| \leq \epsilon$ for all $t \geq t_0$.

Unstable point The $x = 0$ fixed point is unstable if it is not stable.

Asymptotic stability A point $x = 0$ is called asymptotically stable if it is stable and for all t_0 there exists $\delta_0 = \delta_0(t_0)$ such that for all x_0 with $|x_0| \leq \delta_0$

$$\lim_{t\to\infty} x(t; t_0, x_0) = 0.$$

Domain of attraction is the set of all points x_0 for which $x(t; t_0, x_0) \to 0$ as $t \to \infty$.

Attractor is a set with an open neighborhood of points that all approach the set as $t \to \infty$.

Invariant set A set $S \subset P$ is an invariant set for the flow map $F^t : P \to P$ if $F^t(S) = S$ for all $t \in \mathbb{R}$.

3.2 Stability based on linearization

Linearization of system $\dot{x} = f(x)$ at a fixed point p is defined as the system $\dot{y} = Ay$ with $y = x - p \in \mathbb{R}^n$, $A = \mathrm{D}f(p) \in \mathbb{R}^{n\times n}$ by Taylor expanding.

3.3 Review of linear systems

3.4 Stability of fixed points in linear systems

Exponential bound by max eigenvalue For system $\dot{y} = Ay$, there exists a constant $C > 0$ such that

$$|y(t)| = |\varphi(t)y_0| \leq \|\varphi(t)\||y_0| \leq Ce^{\mu t}|y_0|,$$

where $\mu = \nu + \max_j \mathrm{Re}(\lambda_j)$ with $\nu \geq 0$ as small as needed and $\lambda_1, \lambda_2, \ldots$ the eigenvalues of A.

Theorem (Stability of linear systems) Consider the linear system $\dot{y} = Ay$ with $y \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$.

1. Assume that $\mathrm{Re}\lambda_j < 0$ for all j. Then $y = 0$ is asymptotically stable.
2. Assume that $\mathrm{Re}\lambda_j \leq 0$, or equality of geometric and algebraic multiplicities for all eigenvalues λ_k with $\mathrm{Re}\lambda_k = 0$. Then $y = 0$ is stable.
3. Assume that there exists $\mathrm{Re}\lambda_k > 0$. Then $y = 0$ is unstable.

3.5 Stability of fixed points in nonlinear systems

C^k-equivalence Two dynamical systems are C^k-equivalent, $k \in \mathbb{N}$, on an open set $U \subset \mathbb{R}^n$ if there exists a C^k-diffeomorphism $h : U \to U$ that maps orbits of one dynamical systems to orbits of the second, preserving orientation, but not necessarily the parametrization of the orbit by time, i.e. for all $x \in U$ and $t_1 \in \mathbb{R}$ there exists $t_2 \in \mathbb{R}$ such that $h(F^{t_1}(x)) = G^{t_2}(h(x))$.

Topological equivalence is C^k-equiv for $k = 0$.

Trick to check for topological equivalence Check if in both systems the number of eigenvalues of positive (negative, zero) real part matches.

Hyperbolic fixed point The fixed point $x = x_0$ of a nonlinear system $\dot{x} = f(x)$, $f(x_0) = 0$, is called hyperbolic if the eigenvalue of f's linearization, $\mathrm{D}f(x_0)$, satisfy $\mathrm{Re}\lambda_i \neq 0$ for all i.

Linear stability $\sim$ nonlinear stability The linearized stability type of a hyperbolic fixed point is preserved under small perturbations to the nonlinear system.

Hartman-Grobman If the fixed point x_0 of a nonlinear system $\dot{x} = f(x)$, $f(x_0) = 0$ is hyperbolic, then this system is topologically equivalent to it linearization in a neighborhood of x_0.

Hence, for hyperbolic fixed points, linearization predicts the correct stability type and local flow geometry.

Stable/ unstable subspaces

$$E^s = \mathrm{span}\{y_0 : F^t y_0 \to 0 \text{ as } t \to \infty\} \quad \text{(Stable)}$$
$$E^u = \mathrm{span}\{y_0 : F^t y_0 \to 0 \text{ as } t \to -\infty\} \quad \text{(Unstable)}$$

3.6 Determining asymptotic stability from linearization

Hurwitz criterion For a characteristic polynomial $\det(A - \lambda I) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$ with $a_0 > 0$. Then

$$\mathrm{Re}\lambda_i < 0 \iff D_i > 0,$$
$$\mathrm{Re}\lambda_i < 0 \implies a_i > 0$$

for all i and subdeterminants D_i

$$D_n = \begin{pmatrix} a_1 & a_0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & a_{n-1} & 0 \\ 0 & 0 & \ldots & 0 & a_n \end{pmatrix}.$$

3.7 Lypanov's direct (2nd) method for stability

Problems with linear stability analysis

- $\mathrm{Re}\lambda_i = 0$ makes analysis inconclusive
- linearization does not address the size of the domain of stability.

Theorem (Lyapunov (un)stability) Consider the system $\dot{x} = f(x)$ with $f \in C^r, x \in \mathbb{R}^n, f(x_0) = 0$. Assume that there exists a Lyapunov function $V : U \to \mathbb{R}$, $V \in C^1(U), U \subset \mathbb{R}^n$ open, $x_0 \in U$ such that $V(x_0) = 0$, $V(x) > 0$ for all $x \in U \setminus \{x_0\}$.

1. If $\dot{V}(x) = \langle \mathrm{D}V(x), f(x)\rangle \leq 0$ for all $x \in U$, then $x = 0$ is **(Lyapunov) stable**.
2. If $\dot{V}(x) = \langle \mathrm{D}V(x), f(x)\rangle < 0$ for all $x \in U$, then $x = 0$ is asymptotically stable.
3. If $\dot{V}(x) = \langle \mathrm{D}V(x), f(x)\rangle \leq 0$ for all $x \in U$ and the set $\{x \in U \mid \dot{V}(x) = 0\}$ does not contain full trajectories, then $x = 0$ is asymptotically stable.
4. If $\dot{V}(x) = \langle \mathrm{D}V(x), f(x)\rangle > 0$ for all $x \in U$, then $x = 0$ is unstable.

Theorem (Unstability by indefiniteness) Consider the system $\dot{x} = f(x)$ with $f \in C^r, x \in \mathbb{R}^n, f(x_0) = 0$. Assume that there exists an indefinite Lyapunov function $V : U \to \mathbb{R}$, $V \in C^1(U), U \subset \mathbb{R}^n$ open, $x_0 \in U$ such that there exist $x_1, x_2 \in U$ with $V(x_1) > 0, V(x_2) < 0$ and $V(x_0) = 0$. Assume that $\dot{V}$ is definite near x_0. Then x_0 is unstable.

Examples: pendulum, friction pendulum

4 Bifurcations & fixed points

4.1 Local nonlinear dynamics

Consider system $\dot{x} = f(x), f(x) \in C^r, r \geq 1$ with a fixed point p,$f(p) = 0$. The linearized system $\dot{y} = \mathrm{D}_p f$

Stable/ unstable subspaces are the invariant subspaces

$$E^s = \mathrm{span}\{\mathrm{Re}(e_j), \mathrm{Im}(e_j) \mid \mathrm{Re}(\lambda_j) < 0\} \quad \text{(Stable)}$$
$$E^u = \mathrm{span}\{\mathrm{Re}(e_j), \mathrm{Im}(e_j) \mid \mathrm{Re}(\lambda_j) > 0\} \quad \text{(Unstab)}$$
$$E^c = \mathrm{span}\{\mathrm{Re}(e_j), \mathrm{Im}(e_j) \mid \mathrm{Re}(\lambda_j) = 0\} \quad \text{(Center)}$$

Remarks (Hyperbolic fixed points) The fixed point p is hyperbolic iff $E^c = \emptyset$.

Remark (Decay in E^s, E^u) Solutions in E^s (E^u) decay to $y = 0$ as $t \to \infty$ ($t \to -\infty$)

Theorem (Center manifold) There exists

1. a unique **stable manifold** $W^s(x_0)$ s.t
   (a) $W^s(x_0)$ is a C^r surface, tangent to E^s at x_0 with $\dim W^s(x_0) = \dim E^s$,

(b) $W^s(x_0)$ is invariant; for $x \in W^s(x_0)$: $|F^t(x_0)| \leq K \cdot \exp((\max_{\text{Re}(\lambda_j)<0} \text{Re}(\lambda_j)t))$, $t \geq 0$, $|x - p|$ small,

2. a unique **unstable manifold** $W^u(x_0)$ s.t

   (a) $W^s(x_0)$ is a C^r surface, tangent to E^u at x_0 with $\dim W^u(x_0) = \dim E^u$,
   (b) $W^u(x_0)$ is invariant; for $x \in W^u(x_0)$: $|F^t(x_0)| \leq K \cdot \exp((\min_{\text{Re}(\lambda_j)<0} \text{Re}(\lambda_j)t))$, $t \leq 0$, $|x - p|$ small,

3. a **center manifold** $W^c(x_0)$ s.t

   (a) $W^c(x_0)$ is a C^{r-1} surface, tangent to E^c at x_0 with $\dim W^c(x_0) = \dim E^c$,
   (b) $W^s(x_0)$ is invariant.

4.2 The center manifold

4.3 Center manifolds depending on parameters

4.4 Bifurcations

4.5 Codimension-one bifurcations of fixed points

5 Nonlinear dynamical systems on the plane

5.1 One-DOF conservative mechanical systems

Energy is conserved $\frac{\mathrm{d}E(x(t))}{\mathrm{d}t} = \frac{\partial E}{\partial x_1}\dot{x}_1 + \frac{\partial E}{\partial x_2}\dot{x}_2 = 0$.
From Newton's law $\dot{x}_1 = x_2, \dot{x}_2 = -\frac{1}{m}\frac{\mathrm{d}V}{\mathrm{d}x_1}$.
For conservative systems $E(x) = \frac{1}{2}mx_2^2 + V(x_1) = E_0 = const.$ and hence $x_2 = \pm\sqrt{\frac{2}{m}(E_0 - V(x_1))}$.

1. Trajectories form symmetric pairs (w.r.t. the x_1-axis) of the same energy.
2. **Clockwise orientation** for trajectories: $x_2 > 0 \implies x_1$ increases (and vv) due to $\dot{x}_1 = x_2$.
3. **Local minima** in V become center fixed points surrounded by closed orbits.
4. **Local maxima** in V become a saddle-type fixed point.
5. **Heteroclinic orbits** come from a symmetric potential with origin at a local maximum that has an identical local maximum some distance away.
6. **Homoclinic orbits** come from an asymmetric potential with the origin at a local maximum near a local minimum.

5.2 Global behavior in 2d autonomous DS

Consider system $\dot{x} = f(x), x \in \mathbb{R}^2, f \in C^1$. Assume that solutions exist for all times (and hence guarantees uniqueness of solution).

Definition $p \in \mathbb{R}^2$ is a **ω-limit point** of x_0 if there exists a monotone increasing unbounded sequence $\{t_i\}_{i\in\mathbb{N}}, t_i \geq 0$ such that

$$\lim_{i\to\infty} x(t_i, x_0) = p.$$

We denote by $\omega(x_0)$ the **ω-limit set of** x_0 that consists of all ω-limit points of x_0. A point $q \in \mathbb{R}^2$ is a **α-limit point** of x_0 if it is a ω-limit point in backward time and $\alpha(x_0)$ similarly denotes the set of α-limit points of x_0.

Theorem If $x(t; x_0)$ is bounded, then $\omega(x_0), \alpha(x_0)$ are non-empty, closed, connected, and invariant, i.e. consist of full trajectories.

Theorem (Poincare-Bendixson) If $x(t; x_0)$ is bounded, then $\omega(x_0), \alpha(x_0)$ must be one of the following:

1. A connected set of fixed points.
2. A limit cycle.
3. A set of fixed points and their connecting sets homoclinic/ heteroclinic orbits

Remarks/ Consequences

1. Homoclinic/ heteroclinic orbits are generally not robust, i.e. small perturbations make them non-homoclinic/non-heteroclinic. However, for conservative system they are robust.
2. A forward-invariant, bounded open set without fixed points must contain a limit cycle.
3. **Bendixon-criterion**: For $\dot{x} = f(x), x \in \mathbb{R}^2, f \in C^1, U \subset \mathbb{R}^2$ simply connected and $\operatorname{div} f(x) \neq 0$ on U. Then there exist no limit cycles in U.
4. Purely damped or purely forces perturbations of a conservative system cannot have limit cycles.

6 Time dependent dynamical systems

6.1 Nonautonomous linear systems

6.2 Time-periodic homogenous linear systems

6.3 Averaging

Consider the systems, $x \in U \subseteq \mathbb{R}^n$ with $0 \leq \epsilon \ll 1$:

$$\dot{x} = \epsilon f(x, t, \epsilon) \tag{1}$$

$$\dot{y} = \epsilon \overline{f}(y, \epsilon) := \epsilon \frac{1}{T} \int_0^T f(y, t, 0)\, \mathrm{d}t \tag{2}$$

Theorem (Averaging Principle) There exists a C^r change of coordinates $x = z + \epsilon w(z, t, \epsilon)$ under which the system (1) becomes

$$\dot{z} = \epsilon \overline{f}(z) + \mathcal{O}(\epsilon^2).$$

Moreover, if $x(t), y(t)$ are solutions of (1), (2) based at x_0, y_0, respectively, at $t = 0$, and $|x_0 - y_0| = \mathcal{O}(\epsilon)$, then $|x(t) - y(y)| = \mathcal{O}(\epsilon)$ on a time scale $t \sim \frac{1}{\epsilon}$.